Problem 1.16, Holton and Hakim

It is stated that temperature decrease linearly with height and has a constant lapse rate γ. This may be represented as:

$$T(z) = T_0 - \gamma Z$$

where T_0 is the temperature at $z=0$.

Begin derivation with $\frac{dp}{dz} = -\rho g$ and $p = \rho RT$ such that

$$\frac{dp}{p} = -\frac{g}{RT(z)} dZ \implies \frac{dp}{p} = -\frac{g}{R(T_0 - \gamma Z)} dz$$

Integrate from surface values p_0, T_0 to a pressure level p_1 with temperature T_1 at height z

$$\int_{p_0}^{p_1} \frac{dp}{p} = -\frac{g}{R} \int_0^z \frac{1}{T_0 - \gamma z} dz$$

Integrate right side using substitution

$$u = T_0 - \gamma Z$$
$$du = -\gamma dz \quad \therefore\ dz = \frac{-du}{\gamma}$$

$$\int_{p_0}^{p_1} \frac{dp}{p} = +\frac{g}{\gamma R} \int_0^z \frac{1}{u} du$$

$$\ln|p|\Big]_{p_0}^{p_1} = \frac{g}{\gamma R}\ln|u|\Big]_0^z$$

$$\ln|p|\Big]_{p_0}^{p_1} = \frac{g}{\gamma R}\ln|T_0-\gamma z|\Big]_0^z$$

Since $p>0$, and temperature (in Kelvins) >0, can drop, $|p| \to p$, $|T_0-\gamma z| \to T_0-\gamma z$.

Finish integral

$$\ln\frac{p_1}{p_0} = \frac{g}{\gamma R}\ln\frac{T_0-\gamma z}{T_0}$$

Rewrite as

$$\ln\frac{p_1}{p_0} = \ln\frac{T_0-\gamma z}{T_0}^{\frac{g}{\gamma R}}$$

Take antilogs

$$\frac{p_1}{p_0} = \left(\frac{T_0-\gamma z}{T_0}\right)^{\frac{g}{\gamma R}}$$

Rewrite as

$$\left(\frac{p_1}{p_0}\right)^{\frac{\gamma R}{g}} = \left(\frac{T_0-\gamma z}{T_0}\right)$$

Rewrite some more as

$$\left(\frac{p_0}{p_1}\right)^{-\frac{\gamma R}{g}} = 1 - \frac{\gamma z}{T_0}$$

Solve for z

$$\boxed{z = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_0}{p_1}\right)^{-\frac{\gamma R}{g}}\right]}$$