Barotropic Potential Vorticity Equation

Consider the scaled vorticity equation

(11)
$$\frac{D_h(\zeta+f)}{Dt} = -\underbrace{(\zeta+f)}_{\substack{\text{retained} \\ \text{for convenience}}}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Also consider the velocity divergence form of the continuity equation (Eq 2.31, pg 45 in Holton).

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0$$

Let's assume an incompressible fluid such that

$$\frac{D\rho}{Dt} = 0$$

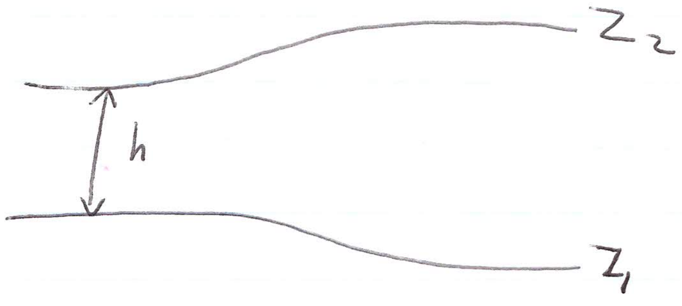
Hence

(12)
$$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Plug (12) into (11)

(13)
$$\frac{D_h(\zeta+f)}{Dt} = (\zeta+f)\frac{\partial w}{\partial z}$$

Now consider an incompressible fluid with variable depth $h(x,y,t) = z_2 - z_1$



Since fluid is barotropic, ζ is constant with height in the layer at all points x, y (thermal wind is zero). Hence we can integrate (13)

$$\frac{D_h(\zeta+f)}{Dt}\underbrace{\int_{z_1}^{z_2} dz}_{h = z_2 - z_1} = (\zeta+f)\int_{w_1}^{w_2} dw$$

or

$$\frac{D_h(\zeta+f)}{Dt}\, h = (w_2 - w_1)(\zeta+f)$$

However, $w = \frac{Dz}{Dt}$, so $w_2 - w_1 = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{Dh}{Dt}$

(14) $$\frac{D_h(\zeta+f)}{Dt} - \left(\frac{1}{h}\frac{Dh}{Dt}\right)(\zeta+f) = 0$$

Divide through by h

$$\frac{1}{h}\frac{D_h(\zeta+f)}{Dt} - \frac{(\zeta+f)}{h^2}\frac{dh}{dt} = \left(h\frac{d(\zeta+f)}{dt} - (\zeta+f)\frac{dh}{dt}\right)/h^2 = 0$$

And from the quotient rule from calculus

$$\frac{a\frac{db}{dx} - b\frac{da}{dx}}{a^2} = \frac{d}{dx}\left(\frac{b}{a}\right)$$

Gives

(15) $$\boxed{\frac{d}{dt}\left(\frac{\zeta+f}{h}\right) = 0}$$ $\therefore$ $\frac{\zeta+f}{h} = \text{constant}$

$(\zeta+f)/h \equiv$ Potential vorticity (PV) for a barotropic fluid

PV is "conserved."

Now consider the case where $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ $\therefore$ $\frac{\partial w}{\partial z} = 0$,
$\therefore$ $h =$ constant; then

(16) $$\boxed{\frac{d}{dt}(\zeta+f) = 0}$$ $\therefore$ $\zeta+f = \text{constant}$

Absolute vorticity is conserved.