

# Classical Rossby wave dispersion equation

Assume a barotropic, non-divergent atmosphere in one dimension (along x axis)

$$\frac{D(\psi+f)}{Dt} = 0$$

Expand total derivative

$$\frac{\partial}{\partial t}(\psi+f) + u \frac{\partial}{\partial x}(\psi+f) + v \frac{\partial}{\partial y}(\psi+f) = 0$$

Make the "Beta-plane" approximation  $f = f_0 + \beta y$  where  $\beta = \frac{\partial f}{\partial y}$   
Also note that  $\frac{\partial f}{\partial t} = 0$  and  $\frac{\partial f}{\partial x} = 0$

$$\frac{\partial \psi}{\partial t} + u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y} + v \beta = 0$$

Linearize as  $u = \bar{u} + u'$ ;  $v = v'$ ;  $\psi = \psi'$   
 $\bar{u}$   
background mean zonal wind

Variations are only along x except for  $\beta, v'$

$$\frac{\partial \psi'}{\partial t} + \bar{u} \frac{\partial \psi'}{\partial x} + v' \beta = 0$$

Note the  $v' = \frac{\partial \psi}{\partial x}$ ,  $\psi' = \frac{\partial \psi'}{\partial x} - \frac{\partial \psi}{\partial y} = \frac{\partial^2 \psi'}{\partial x^2}$

no variation  
of  $v'$  along  $y$

$$\textcircled{\text{I}} \quad \frac{\partial}{\partial t} \left( \frac{\partial^2 \psi}{\partial x^2} \right) + \bar{u} \frac{\partial}{\partial x} \left( \frac{\partial^2 \psi}{\partial x^2} \right) + \frac{\partial \psi}{\partial x} B = 0$$

Assume a solution of  $\psi = \hat{\psi} \exp[i(kx - vt)]$

write as  $Q$  to shorten notation

Since  $\frac{\partial \psi}{\partial x} = ik \hat{\psi} \exp[Q]$   $\textcircled{\text{II}}$  and  $\frac{\partial^2 \psi}{\partial x^2} = i^2 k^2 \hat{\psi} \exp[Q]$

and then  $\rightarrow \frac{\partial}{\partial t} \left( \frac{\partial^2 \psi}{\partial x^2} \right) = i v k^2 \hat{\psi} \exp[Q]$   $\textcircled{\text{III}}$

(Note that another  
negative from  $i$  of  
 $\exp[i(kx - vt)]$  creates another negative  
sign

and  $\rightarrow \frac{\partial}{\partial x} \left( \frac{\partial^2 \psi}{\partial x^2} \right) = -ik^3 \hat{\psi} \exp[Q]$   $\textcircled{\text{IV}}$

Plug  $\textcircled{\text{II}}$ ,  $\textcircled{\text{III}}$  and  $\textcircled{\text{IV}}$  into  $\textcircled{\text{I}}$

$$i v k^2 \hat{\psi} \exp[Q] - \bar{u} i k^3 \hat{\psi} \exp[Q] + i k \hat{\psi} \exp[Q] B = 0$$

Note that  $i$  cancels out, a  $k$  cancels out, and  $\hat{\psi} \exp[Q]$  cancels out

$$\nu k - \bar{u} k^2 + \beta = 0$$

Solve for  $\nu$  to get dispersion relationship

$$\nu = \bar{u} k - \frac{\beta}{k}$$

The phase speed is  $c = \frac{\nu}{k}$

$$c = \bar{u} - \frac{\beta}{k^2}$$

Since  $k = \frac{2\pi}{L}$

$$c = \bar{u} - \frac{\beta L^2}{4\pi^2}$$

## Implications

- 1) If  $\bar{U} = 0$ , the phase speed of a Rossby wave is negative. They move west, which in meteorology parlance we also call "retrograde." This is also known as the "Beta effect."
- 2) If  $\bar{U} = 0$ , the longer synoptic waves retrograde faster
- 3) For a background  $\bar{U}$  west wind, usually  $\bar{u} > \frac{\beta L^2}{4\pi^2}$ , and synoptic waves move east, The shorter waves move faster to the east. However, for really long waves,  $\bar{u} < \frac{\beta L^2}{4\pi^2}$ , and such waves will still retrograde.

Group velocity for Rossby waves is determined as

$$c_g = \frac{\partial \omega}{\partial k}$$

$$c_g = \frac{\partial}{\partial k} (\bar{u}k - \beta k^{-1})$$

$$= \bar{u} - (-1)\beta k^{-2}$$

$$c_g = \bar{u} + \frac{\beta}{k^2}$$

Note that, even if  $\bar{u} = 0$ , the group velocity of Rossby waves is to the east. This is opposite of their phase speed when  $\bar{u} = 0$ !