# Curvature and vectors

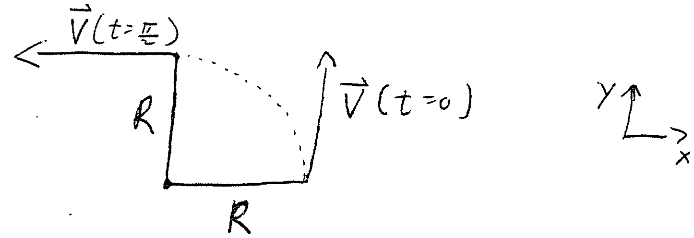
Consider the position vector $\vec{r}(x,y,t)$ along a circle with a radius of curvature $R$:

(1) $$\vec{r} = R\cos t\,\hat{i} + R\sin t\,\hat{j}$$

The velocity vector is then

(2) $$\vec{V} = -R\sin t\,\hat{i} + R\cos t\,\hat{j}$$

At $t=0$, $\vec{V} = r\hat{j}$, at $t=\frac{\pi}{2}$, $\vec{V} = -r\hat{i}$



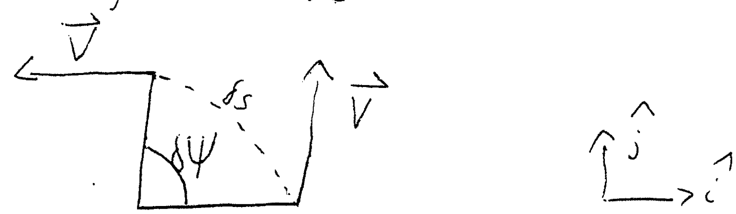

The distance traveled is represented by the dashed line. The length of this curve, $s(x,y,t)$, is the integral of the length of the velocity vector in that time increment

(3) $$\delta s = \int |\vec{V}|\,dt$$

From (3), it follows that $|\vec{V}| = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$

(4) $$\boxed{|\vec{V}| = \frac{ds}{dt}}$$

We can also measure the rate at which $\vec{V}$ turns as it moves along the curve




by measuring the change in $\Psi$ (the direction angle that $\vec{V}$ makes with $\hat{i}$) with respect to $s$. At each point, the value of $\frac{d\Psi}{ds}$, measured in radians per unit of length along the curve, is called the curvature. The usual notation is $K$.

(5) $$\boxed{\text{Curvature} \equiv K = \frac{d\Psi}{ds}}$$

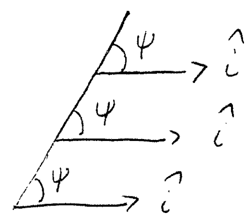
By referring to any standard calculus book, $K$ may be computed as (Thomas and Finney, pg 772),

(6) $$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

(7) $$K = \frac{\left|\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)\right|}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}$$

Example: Show that the curvature of a straight line is zero

Solution: On a straight line, $\psi$ has a constant value




Therefore, $\frac{d\psi}{ds} = 0$ and $K = \frac{d\psi}{ds} = 0$

Example: Show that the curvature of a circle is $R^{-1}$.

Solution: From (2)

$$\frac{dx}{dt} = -R\sin t \qquad \frac{dy}{dt} = R\cos t$$

$$\frac{d^2x}{dt^2} = -R\cos t \qquad \frac{d^2y}{dt^2} = -R\sin t$$

$$K = \frac{\left|(-R\sin t)(-R\sin t) - (R\cos t)(-R\cos t)\right|}{\left[R^2\sin^2 t + R^2\cos^2 t\right]^{\frac{3}{2}}} = \frac{R^2}{R^3}$$

(8) $$\boxed{K = \frac{1}{R}}$$

For a straight line, $R \to \infty$

# Unit tangent and normal vectors

The unit tangent vector to the position vector $\vec{r}(x, y, t)$ is

(9) $$\boxed{\hat{\tilde{\iota}} = \frac{d\vec{r}}{ds}}$$

where $\hat{\tilde{\iota}}$ is oriented parallel to the horizontal velocity trajectory at each point.

Given this definition, $\vec{V}$ may be defined as

(10) $$\boxed{\vec{V} = |\vec{V}|\,\hat{\tilde{\iota}}}$$

From (2) $\hat{\tilde{\iota}}$ may be computed along a circle as

$$\hat{\tilde{\iota}} = \frac{\vec{V}}{|\vec{V}|}$$

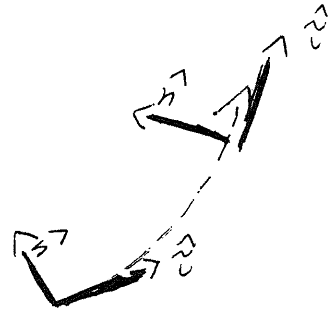
$$|\vec{V}| = \sqrt{(R\sin t)^2 + (R\cos t)^2} = \sqrt{R^2(\sin^2 t + \cos^2 t)} = R$$

There fore

(11) $$\hat{\tilde{\iota}} = \frac{\vec{V}}{|\vec{V}|} = \frac{-R\sin t\,\hat{\imath} + R\cos t\,\hat{\jmath}}{R} = -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}$$

If one is going to define a coordinate parallel to $\vec{V}$, it makes sense to also define a coordinate perpendicular to $\vec{V}$ at each point.

Therefore, let's define $\hat{n}$ to be oriented normal to $\vec{V}$ at each point. Let's further define $\hat{n}$ as positive left of flow direction.




Can we relate $\hat{n}$ to $\hat{\tau}$ ?

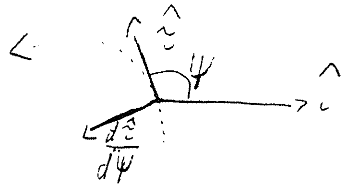
$\hat{\tau}$ may be written in terms of $\psi$ (consistent with (11))

$$\hat{\tau} = \hat{i}\cos\psi + \hat{j}\sin\psi$$

Therefore

$$\frac{d\hat{\tau}}{d\psi} = -\hat{i}\sin\psi + \hat{j}\cos\psi = \hat{i}\cos\left(\psi + \frac{\pi}{2}\right) + \hat{j}\sin\left(\psi + \frac{\pi}{2}\right)$$

from which it follows that $\frac{d\hat{\tau}}{d\psi}$ is the unit vector obtained by rotating $\hat{\tau}$ counterclockwise through $\frac{\pi}{2}$ radians. Thus, $\frac{d\hat{\tau}}{d\psi}$ is normal to the curve at all times.



From the chain rule, $\frac{d\hat{\tilde{i}}}{ds} = \frac{d\hat{\tilde{i}}}{d\psi}\frac{d\psi}{ds}$, therefore its magnitude is

$$\left|\frac{d\hat{\tilde{i}}}{ds}\right| = \left|\frac{d\hat{\tilde{i}}}{d\psi}\right|\left|\frac{d\psi}{ds}\right| = (1)(K) = K$$

The vector $\frac{d\hat{\tilde{i}}}{ds}$ is normal to the curve because it is a scalar multiple of $\frac{d\hat{\tilde{i}}}{d\psi}$

The unit vector $\hat{n}$ is obtained by

(12) $$\boxed{\hat{n} = \frac{\left(\frac{d\hat{\tilde{i}}}{ds}\right)}{\left|\frac{d\hat{\tilde{i}}}{ds}\right|} = \frac{1}{K}\frac{d\hat{\tilde{i}}}{ds}}$$ (Thomas and Finney, pg 775)

$\hat{n}$ may be rewritten as

$$\hat{n} = R\,\frac{d\hat{\tilde{i}}/dt}{ds/dt} = R\,\frac{d\hat{\tilde{i}}/dt}{|\vec{V}|}$$

or

(13) $$\boxed{\frac{d\hat{\tilde{i}}}{dt} = \frac{|\vec{V}|}{R}\,\hat{n}}$$

# Equations of motion in natural coordinates

Since $\vec{V} = |\vec{V}|\hat{\tau}$ (Eq 10)

(14) $$\frac{D\vec{V}}{Dt} = \hat{\tau}\frac{D|\vec{V}|}{Dt} + |\vec{V}|\underbrace{\frac{D\hat{\tau}}{Dt}}_{\frac{|\vec{V}|}{R}\hat{n}}$$

from (13)

or

(15) $$\frac{D\vec{V}}{Dt} = \hat{\tau}\frac{D|\vec{V}|}{Dt} + \hat{n}\frac{|\vec{V}|^2}{R}$$

And

$$\frac{D\vec{V}}{Dt} = -f\hat{k}\times\vec{V} - \nabla\Phi$$

The Coriolis force always acts normal to the direction of motion, thus

(16) $$-f\hat{k}\times\vec{V} = -f|\vec{V}|\hat{n}$$

While the pressure gradient force is

(17) $$-\nabla\Phi = -\left(\hat{\tau}\frac{\partial\Phi}{\partial s} + \hat{n}\frac{\partial\Phi}{\partial n}\right)$$

Hence

$$(18)\quad \frac{D\vec{V}}{Dt} = -f|\vec{V}|\hat{n} - \hat{t}\frac{\partial \Phi}{\partial s} - \hat{n}\frac{\partial \Phi}{\partial n} = \hat{t}\frac{D|\vec{V}|}{Dt} + \hat{n}\frac{|\vec{V}|^2}{R}$$

Therefore, the horizontal momentum equations may be written into the following component equations in natural coordinates

$$\boxed{\begin{aligned} &(19)\quad \frac{D|\vec{V}|}{Dt} = -\frac{\partial \Phi}{\partial s} \\ &(20)\quad \frac{|\vec{V}|^2}{R} + f|\vec{V}| = -\frac{\partial \Phi}{\partial n} \end{aligned}}$$

Note: $R > 0$ if air parcels turn toward the left following motion (cyclonic)

$R < 0$ if air parcels turn toward the right following motion (anticyclonic).

## Geostrophic Flow in natural coordinates

For steady state $\left(\frac{D|\vec{V}|}{Dt} = 0\right)$, and

for straight line flow $(R \to \infty)$, (2.0)

becomes

(21) $$\boxed{|\vec{V}_g| = -\frac{1}{f}\frac{\partial \Phi}{\partial n}}$$ see Fig 3.2

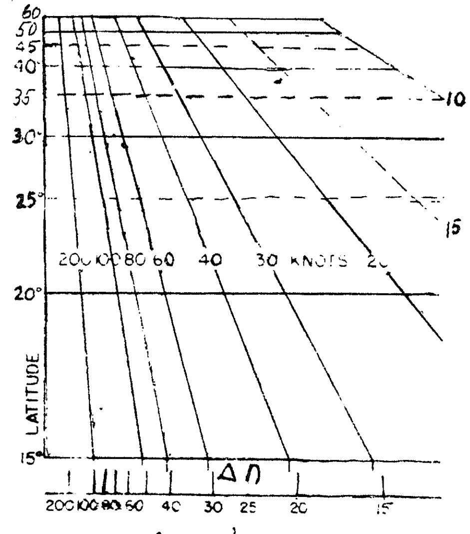
~~Inertial Flow~~

## Geostrophic wind scale

Since geopotential height is typically contoured in 60 m intervals, a graph may be constructed for operational use:

Geostrophic Windscale

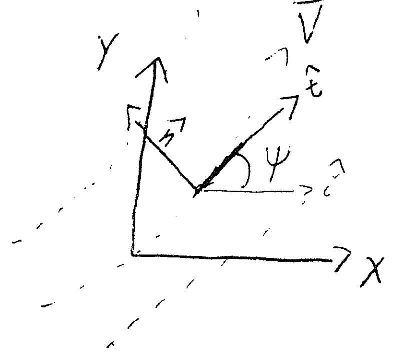
$\phi$



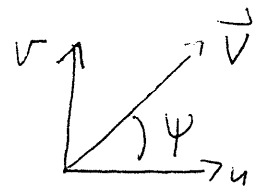

$m\,s^{-1}$

# Horizontal Divergence in Natural Coordinates

Consider a set of streamlines with a horizontally varying windspeed field (implying contours of isotachs, but not shown to avoid messier drawing).




The vector $\vec{V}$ may be broken into its components u and v




$$u = |\vec{V}| \cos\psi$$
$$v = |\vec{V}| \sin\psi$$

Therefore

$$\frac{\partial u}{\partial x} = \cos\psi \frac{\partial |\vec{V}|}{\partial x} - |\vec{V}| \sin\psi \frac{\partial \psi}{\partial x}$$

$$\frac{\partial v}{\partial y} = \sin\psi \frac{\partial |\vec{V}|}{\partial y} + |\vec{V}| \cos\psi \frac{\partial \psi}{\partial y}$$

Now rotate x, y axes so that $\psi \to 0$. Then $\cos\psi \to 1$ and $\sin\psi \to 0$. Also $x \to s$ and $y \to n$

$$(22) \quad \boxed{\nabla_h \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \underbrace{\frac{\partial |\vec{V}|}{\partial s}}_{\text{Stretching term}} + \underbrace{|\vec{V}| \frac{\partial \psi}{\partial n}}_{\text{Diffluence term}}}$$

; Stretching $\Rightarrow$ speed divergence

Diffluence $\Rightarrow$ Directional Divergence

Recall that $\frac{1}{R} = K = \frac{d\psi}{ds}$ (Eq 5).
Likewise, we can define a "radius of curvature" for the line normal to $s(x,y,t)$

(23) $$\frac{1}{R_N} = K_N = \frac{d\psi}{dn}$$

Hence, horizontal divergence may be written

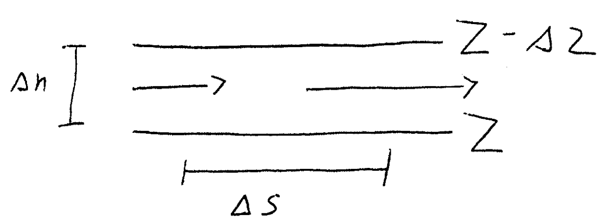
(24) $$\boxed{\nabla_h \cdot \vec{V} = \frac{\partial |\vec{V}|}{\partial s} + \frac{|\vec{V}|}{R_N}}$$

Interpretation:

The stretching term measures the change in $|\vec{V}|$ along $s(x,y,t)$. Practically speaking, we access how $|\vec{V}|$ is changing along streamlines or geopotential heights.

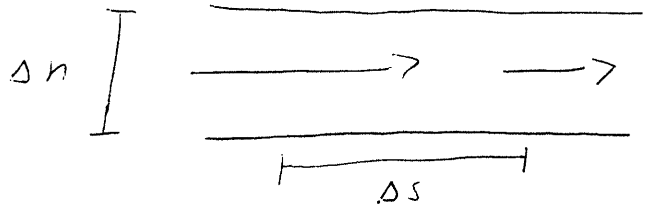
---

Hypothetical Example 1; $\Delta n$ = constant, but $|\vec{V}|$ changes downstream




$\frac{\partial |\vec{V}|}{\partial s} > 0$, since $|\vec{V}|$ increases downstream, the atmosphere is being "stretched." More air is being removed than added. This is <u>speed</u> <u>divergence</u>.

Likewise

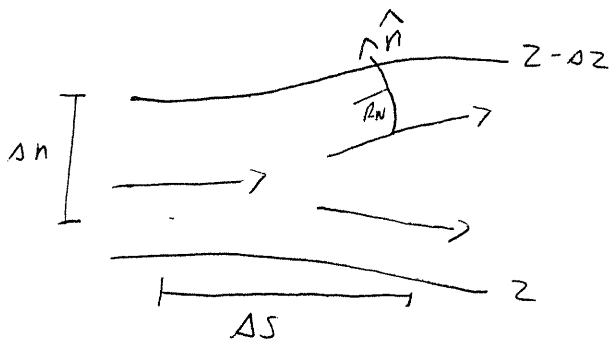



$\frac{d|\vec{V}|}{ds} < 0$, speed convergence

Often, speed divergence and convergence is accessed by looking at isotach field.

---

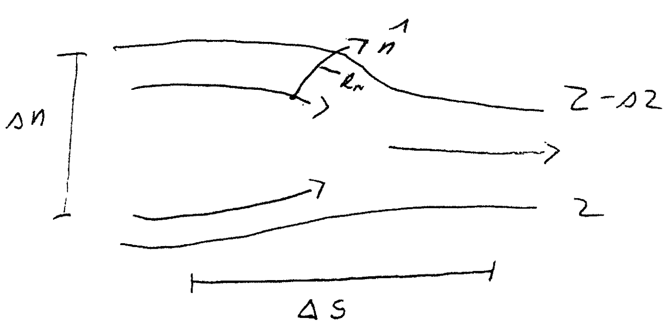
Example 2; $\Delta S$ = constant, but $\Delta n$ changes downstream with $|\vec{V}|$ = constant.




Since $R_N > 0$; directional divergence, also called diffluence.

Air is "spreading out" at constant speed.

Likewise




$R_N < 0$

directional convergence, also called confluence.

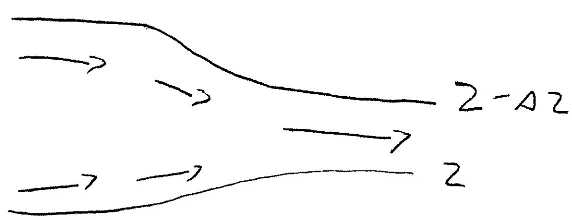
NOTE! confluence is not total convergence!

To see if $\nabla_h \cdot \vec{V} > 0$ or $\nabla_h \cdot \vec{V} < 0$, need to consider speed divergence (convergence) and diffluence (confluence) together.

Furthermore, terms tend to cancel, making qualitative assessment difficult. Also, $\nabla_h \cdot \vec{V}$ is sensitive to measurement errors, making it difficult to calculate.

---

Example 3:




We have directional convergence, but speed divergence! This is typical. If the pressure gradient tightens downstream, that's associated with $\frac{\partial |\vec{V}|}{\partial s} > 0$ and confluence. So what is $\nabla_h \cdot \vec{V}$?

Determining which is dominating can be hard.

This is why divergence is a small term.

$$\nabla_h \cdot \vec{V} \sim \text{Order of } 10^{-6}\,s^{-1}$$