Arrays, Matrices and Determinants

Spreadsheet calculations lend themselves almost automatically to the use of arrays of values. Arrays in Excel can be either one- or two-dimensional. For the solution of many types of problem, it is convenient to manipulate an entire rectangular array of values as a unit. Such an array is termed a *matrix*. (In Excel, the terms "range," "array" and "matrix" are virtually interchangeable.) An $m \times n$ matrix (m rows and n columns) of values is illustrated below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The values comprising the array are called *matrix elements*. Mathematical operations on matrices have their own special rules, to be discussed in the following sections.

Some Types of Matrices

A matrix which contains a single column of m rows or a single row of n columns is called a *vector*.

A *square matrix* has the same number of rows and columns. The set of elements a_{ij} for which $i = j$ ($a_{11}, a_{22}, \ldots, a_{nn}$) is called the *main diagonal* or *principal diagonal*.

If all the elements of a square matrix are zero except those on the main diagonal, the matrix is termed a *diagonal matrix*. A diagonal matrix whose diagonal elements are all 1 is a *unit matrix*.

An upper triangular matrix has values on the main diagonal and above, but the values of all elements below the main diagonal are zero; similarly, a lower triangular matrix has zero values for all elements above the main diagonal.

A *tridiagonal matrix* contains all zeros except on the main diagonal and the two adjacent diagonals.

A *symmetric* matrix is a square matrix in which $a_{ij} = a_{ji}$.

A *determinant* is a property of a square matrix; there is a procedure for the numerical evaluation of a determinant, so that an $N \times N$ matrix can be reduced to a single numerical value. The value of the determinant has properties that make it useful in certain tests and equations. (See, for example, "Cramer's Rule" in Chapter 9.)

An Introduction to Matrix Mathematics

Matrix algebra provides a powerful method for the manipulation of sets of numbers. Many mathematical operations, such as addition, subtraction, multiplication and division, have their counterparts in matrix algebra. Our discussion will be limited to the manipulations of square matrices. For purposes of illustration, two 3×3 matrices will be defined, namely

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 4 & 3 & 7 \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

Addition or Subtraction. The following examples illustrate addition or subtraction.

Addition of a constant: $$\mathbf{A} + q = \begin{bmatrix} a+q & b+q & c+q \\ d+q & e+q & f+q \\ g+q & h+q & i+q \end{bmatrix}$$

Addition of two matrices (both must have the same dimensions, i.e., contain the same numbers of rows and columns):

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a+r & b+s & c+t \\ d+u & e+v & f+w \\ g+x & h+y & i+z \end{bmatrix}$$

Multiplication or Division. Multiplication or division by a constant:

$$q\mathbf{A} = \begin{bmatrix} qa & qb & qc \\ qd & qe & qf \\ qg & qh & qi \end{bmatrix}$$

Multiplication of two matrices can be either *scalar* or *matrix* multiplication. Scalar multiplication of two matrices consists of multiplying the elements of a matrix by a constant, as shown above, or multiplying corresponding elements of two matrices:

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} a \times r & b \times s & c \times t \\ d \times u & e \times v & f \times w \\ g \times x & h \times y & i \times z \end{bmatrix}$$

Thus it's clear that both matrices must have the same dimensions $m \times n$. Scalar multiplication is commutative, that is, $\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A}$.

Matrix Multiplication. The matrix multiplication of two matrices is somewhat more complicated. The individual matrix elements of the matrix product **C** of two matrices **A** and **B** are

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

where i is the row number and j is the column number. Thus, for example,

$$\mathbf{A} \cdot \mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix} = \begin{bmatrix} ar + bu + cx & as + bv + cy & at + bw + cz \\ dr + eu + fx & ds + ev + fy & dt + ew + fz \\ gr + hu + ix & gs + hv + iy & gt + hw + iz \end{bmatrix}$$

Matrix multiplication is not generally commutative, that is $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$.

Transposition. The transpose of matrix **A**, most commonly written as $\mathbf{A}^{\mathrm{T}}$, is the matrix obtained by exchanging the rows and columns of A; that is, the matrix element a_{ij} becomes the element a_{ji} in the transposed matrix. The transpose of a matrix of N rows and M columns is a matrix of M rows and N columns.

Matrix Inversion. The process of *matrix inversion* is analogous to obtaining the reciprocal of a number a. The matrix relationship that corresponds to the algebraic relationship $a \times (1/a) = 1$ is

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$$

where $\mathbf{A}^{-1}$ is the inverse matrix and **I** is the unit matrix. The process for manual calculation of the inverse of a matrix is complicated and need not be described here, since matrix inversion can be done conveniently using Excel's worksheet function MINVERSE.

Evaluation of the Determinant. A determinant is a mathematical value that can be calculated for a square matrix. Determinants are useful for the solution of systems of simultaneous equations, as will be discussed in chapter 9. The "pencil-and-paper" evaluation of the determinant of a matrix of N rows $\times$ N columns is tedious, but it can be done simply by using Excel's worksheet function MDETERM.

Excel's Built-in Matrix Functions

Performing matrix mathematics with Excel is very simple. Let's begin by assuming that the matrices **A** and **B** have been defined by selecting the 3R × 3C arrays of cells containing the values shown in Figure 3-1 and naming them by using **Define Name**. Remember, we're simply assigning a range name to a range of cells. We usually refer to it as a range or an array; the fact that we are calling it a matrix simply indicates what we intend to do with it.

	B	C	D
3		Matrix A	
4	2	3	4
5	3	2	-1
6	4	3	7

	F	G	H
3		Matrix B	
4	2	0	2
5	0	3	-3
6	-3	-2	1

Figure 3-1. Ranges of cells defined as A and B.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

Addition or Subtraction. To add a constant (e.g., 3) to matrix **A**, simply select a range of cells the same size as the matrix, enter the formula =A+3, then press COMMAND+RETURN or CONTROL+SHIFT+RETURN (Macintosh) or CONTROL+SHIFT+ENTER (Windows). When you "array-enter" a formula by pressing e.g., CONTROL+SHIFT+ENTER, Excel puts braces around the formula, as shown below:

{=A+3}

Do not type the braces; if you do, the result will not be recognized by Excel as a formula.

	D	E	F
8	5	6	7
9	6	5	2
10	7	6	10

Figure 3-2. Result matrix {A + 3}.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

Subtraction of a constant, multiplication or division by a constant, or addition of two matrices is performed in the same way by using standard Excel algebraic operators.

Scalar Multiplication. Scalar multiplication can be either multiplication of the elements of a matrix by a constant, e.g., a formula such as {=3*A}, or multiplication of corresponding elements of two matrices, e.g., {=A*B}. The result of the latter formula is shown in Figure 3-3.

	D	E	F
16	4	0	8
17	0	6	3
18	-12	-6	7

Figure 3-3. Result matrix {**A** × **B**}.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

Matrix multiplication can be accomplished easily by the use of Excel's worksheet function MMULT(***matrix1, matrix2***). For the matrices **A** and **B** defined above, entering the formula =MMULT(A,B) yields the result shown in Figure 3-4 while the formula =MMULT(B,A) yields the result shown in Figure 3-5.

	D	E	F
24	-8	1	-1
25	9	8	-1
26	-13	-5	6

Figure 3-4. Result matrix **A**·**B**.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

	D	E	F
28	12	12	22
29	-3	-3	-24
30	-8	-10	-3

Figure 3-5. Result matrix **B·A**.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

Matrix multiplication of two matrices is possible only if the matrices are *conformable*, that is, if the number of columns of **A** is equal to the number of rows of **B**. The opposite condition, if the number of *rows* of **A** is equal to the number of *columns* of **B**, is not equivalent. The following examples, involving multiplication of a matrix and a vector, illustrate the possibilities:

MMULT (4 × 3 matrix, 3 × 1 vector) = 3 × 1 result vector

MMULT (4 × 3 matrix, 1 × 4 vector) = #VALUE!

MMULT (1 × 4 vector, 4 × 3 matrix) = 1 × 4 result vector

In other words, the two inner indices must be the same.

Transposition. The *transpose* of a matrix may be calculated by using the worksheet function TRANSPOSE(***array***) or obtained manually by using the Transpose option in the **Paste Special...** menu command.

The size of the array that can be transposed is limited only by the size of the Excel spreadsheet; the number of rows or columns cannot be greater than 256.

Matrix Inversion. The process for inverting a matrix "manually" (i.e., using pencil, paper and calculator) is complicated, but the operation can be carried out readily by using Excel's worksheet function MINVERSE(***array***). The inverse of the matrix **B** above is shown in Figure 3-6.

	D	E	F
36	-0.25	-0.33333333	-0.5
37	0.75	0.66666667	0.5
38	0.75	0.33333333	0.5

Figure 3-6. Result matrix $\mathbf{B}^{-1}$.
(folder 'Chapter 03 (Matrices) Examples, workbook 'Matrix Math', sheet 'Sheet1')

The size of the matrix must not exceed 52 rows by 52 columns.

Evaluation of the Determinant. The determinant of a matrix of *N* rows × *N* columns can be obtained by using the worksheet function MDETERM(***array***).

The function returns a single numerical value, not an array, and thus you do not have to use CONTROL+SHIFT+ENTER. The value of the determinant of **B**, represented by |**B**|, is 12.