<u>Synoptic Meteorology I: Finite Differences</u>

16-18 September 2014

Partial Derivatives (or, Why Do We Care About Finite Differences?)

With the exception of the ideal gas law, the equations that govern the evolution of fundamental atmospheric properties such as wind, pressure, and temperature (known as the *primitive equations*) are fundamentally reliant upon partial derivatives. Indeed, many thermodynamic and kinematic properties of the atmosphere are typically expressed in terms of partial derivatives. We will explore many specific examples of such equations throughout both this and next semester.

Mathematically speaking, the partial derivative of some generic field f with respect to some generic variable x can be expressed as:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \tag{1}$$

In other words, $\frac{\partial f}{\partial x}$ is equal to the value of $\frac{\Delta f}{\Delta x}$ as Δx approaches (but does not equal) zero. Thus, for small (or finite) values of Δx, we can approximate $\frac{\partial f}{\partial x}$ by $\frac{\Delta f}{\Delta x}$. That begs the question: how do we compute $\frac{\Delta f}{\Delta x}$ from available atmospheric data?

To do so, we use what are known as *finite differences* to <u>approximate</u> the value of Δf over some finite Δx. Applied to isoplethed analyses of meteorological fields, finite differences enable us to evaluate the sign and/or magnitude of a given quantity that depends upon one or more partial derivatives. Applied to gridded data, such as is used and produced by numerical weather prediction models, finite differences are one means by which the primitive equations can be solved so as to obtain a numerical weather forecast. In the following, we wish to describe how finite difference approximations are obtained, the degree to which each is an approximation, and begin to describe how they can be applied to the atmosphere.

Developing Finite Difference Approximations

First, let us consider a generic continuous function $f(x)$, a graphical example of which is depicted below in Figure 1. This function doesn't necessarily represent a meteorological field, but it doesn't not necessarily represent one either; it is simply a generic function. Along the curve given by $f(x)$, there are three points of interest: x_a, x_{a+1}, and x_{a-1}. The function $f(x)$ has the values $f(x_a)$, $f(x_{a+1})$, and $f(x_{a-1})$ at these three points, respectively. The distance between x_a and x_{a-1} is equal to the distance between x_a and x_{a+1}, and we can denote this distance as Δx.

The Taylor series expansion of $f(x)$ about $x = b$, where b is some generic point, is given by:

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2!}(x-b)^2 + \frac{f'''(b)}{3!}(x-b)^3 + \ldots \qquad (2)$$

In other words, $f(x)$ is equal to the value of $f(x)$ at $x = b$ plus a series of higher-order terms, each of which has a different derivative (primes), exponent on $x - b$, and factorial (!) order.

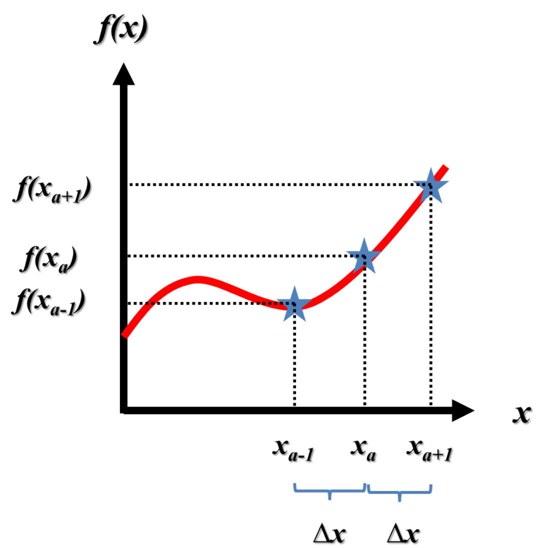

Figure 1. Graphical depiction of a generic function $f(x)$ evaluated at three points. Please see the text for further details.

Let us consider the case where $x = x_{a+1}$ and $b = x_a$. The distance $x - b$, or $x_{a+1} - x_a$, is equal to Δx. Conversely, let us consider the case where $x = x_{a-1}$ and $b = x_a$. The distance $x - b$, or $x_{a+1} - x_a$, is equal to $-\Delta x$. Making use of this information, we can expand (2) for each of these two cases:

$$f(x_{a+1}) = f(x_a) + f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 + \frac{f'''(x_a)}{3!}(\Delta x)^3 + \ldots \qquad (3)$$

$$f(x_{a-1}) = f(x_a) - f'(x_a)\Delta x + \frac{f''(x_a)}{2!}(\Delta x)^2 - \frac{f'''(x_a)}{3!}(\Delta x)^3 + \ldots \qquad (4)$$

Note the similar appearance of (3) and (4) apart from the leading negative signs on the first and third order terms in (4). These arise because $x - b = -\Delta x$ here, as noted above.

From (3) and (4), we are interested in the value of $f'(x_a)$. This is equivalent to $\frac{\partial f}{\partial x}$. We can use (3) and (4) to obtain an expression for this term; we simply need to subtract (4) from (3). Doing so, we obtain the following:

$$f(x_{a+1}) - f(x_{a-1}) = 2f'(x_a)\Delta x + \frac{2f'''(x_a)}{3!}(\Delta x)^3 + \ldots \text{(odd order terms)} \ldots \qquad (5)$$

Note how the zeroth and second order terms in (3) and (4) cancel out in this operation. If we rearrange (5) and solve for $f'(x_a)$, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \ldots \qquad (6)$$

At this point, we wish to neglect all terms higher than the first order term from (6). Doing so, we are left with:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_{a-1})}{2\Delta x} \qquad (7)$$

Equation (7) is what is known as a *centered finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_{a+1}$, subtracting from it the value of f at $x = x_{a-1}$, and dividing the result by the distance between the two points ($2\Delta x$). Note that x here and in later examples can be any variable; it does not have to represent the x-axis or the east-west direction. Equation (7) is equivalent if x is replaced by y, z, p, or any number of other variables.

There exist other ways for us to use (3) and (4) to get expressions for $f'(x_a)$. For instance, we can solve (3) for this term. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} - \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 - \ldots \qquad (8)$$

Neglecting all terms higher than the first order term in (8), we obtain:

$$f'(x_a) = \frac{f(x_{a+1}) - f(x_a)}{\Delta x} \qquad (9)$$

Equation (9) is what is known as a *forward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_{a+1}$, subtracting from it the value of f at $x = x_a$, and dividing the result by the distance between the two points (Δx).

Alternatively, we can solve (4) for $f'(x_a)$. If we do so, we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} + \frac{f''(x_a)}{2!}(\Delta x) - \frac{f'''(x_a)}{3!}(\Delta x)^2 + \ldots \qquad (10)$$

Neglecting all terms higher than the first order term in (10), we obtain:

$$f'(x_a) = \frac{f(x_a) - f(x_{a-1})}{\Delta x} \tag{11}$$

Equation (11) is what is known as a *backward finite difference*. It provides a means of calculating $\frac{\partial f}{\partial x}$ at $x = x_a$ by taking the value of f at $x = x_a$, subtracting from it the value of f at $x = x_{a-1}$, and dividing the result by the distance between the two points (Δx).

Finite Differences as Approximations

Note that we do not necessarily need to neglect the higher-order terms in obtaining any of the above expressions for $f'(x_a)$; we have done so here primarily for simplicity. If we were to retain the higher order terms, we would end up with more accurate approximations for $f'(x_a)$. This highlights a key point: all finite differences are *approximations*. All finite differences are associated with what is known as *truncation error*, which is determined by the power of Δx on the first term that is neglected in obtaining the finite difference approximation.

For instance, consider our centered finite difference given by Equation (7). In obtaining (7), the first term that we neglected in (6) included a $(\Delta x)^2$ term. As a result, we say this finite difference is "second-order accurate." By contrast, consider our forward and backward finite differences, given by Equations (9) and (11), respectively. In obtaining each equation, the first terms that we neglected in (8) and (10) included a (Δx) term. As a result, we say that these finite differences are "first-order accurate." The higher the order of accuracy, the more accurate the finite difference.

In synoptic meteorology, where exact values for partial derivatives are often not necessary, we typically utilize the centered finite difference. Forward and backward finite differences are rarely utilized except along the edges of the data, where the -1 and +1 points may not exist. Higher-order finite differences, typically fourth- or sixth-order accurate, are necessary for numerical weather prediction models given chaos theory, which states that very small differences in data can lead to very large forecast differences.

A Finite Difference Approximation for Second Derivatives

While the first partial derivative of some field provides a measure of its *slope*, sometimes we are interested in evaluating the second partial derivative of some field. Recall from calculus that the second partial derivative of a field provides a measure of its *concavity*; positive second partial derivatives infer that a field is concave up (or convex), while negative second partial derivatives infer that a field is concave down. We can obtain a finite difference approximation for the second partial derivative by adding (3) and (4). Doing so, we obtain:

$$f(x_{a+1}) + f(x_{a-1}) = 2f(x_a) + 2\frac{f''(x_a)}{2!}(\Delta x)^2 + ... \tag{12}$$

If we solve (12) for $f''(x_a)$, we obtain:

$$f''(x_a) = \frac{f(x_{a+1}) + f(x_{a-1}) - 2f(x_a)}{(\Delta x)^2} \tag{13}$$

Equation (13) provides a fourth-order accurate means of evaluating $\frac{\partial^2 f}{\partial x^2}$, or $f''(x_a)$, by adding the value of f at x_{a+1} to the value of f at x_{a-1}, subtracting two times the value of f at x_a, and dividing the result by the square of the distance between points $(\Delta x)^2$.

Just as for the finite difference approximation for the first partial derivative, x here and in later examples can be any variable; it does not have to represent the x-axis or the east-west direction. Equation (13) is equivalent if x is replaced by y, z, p, or any number of other variables. Likewise, just as for the finite difference approximate for the first partial derivative, higher-order accurate finite difference approximations for the second partial derivative are possible if additional terms are not truncated.

Applying Finite Differences: An Example

One of the most important attributes of the wind is its ability to transport. The transport of some quantity by the wind is known as *advection*. We are most often interested in its horizontal transport, or *horizontal advection*, where the horizontal surface can be taken to be Earth's surface, a constant height surface, an isobaric surface, or even an isentropic surface. For convenience, we sometimes refer to horizontal advection simply as advection.

In synoptic meteorology, we are particularly interested in *temperature advection*, referring to the horizontal transport of energy (recall that temperature is simply a measure of the average kinetic energy of the air) by the wind. Patterns of cold air advection and warm air advection reflect the (horizontal) motion of air masses and, as we will see next semester, play a crucial role in forcing vertical motions, can bring about changes in the amplitude of troughs and ridges, and can influence cyclone and anticyclone development.

Mathematically, temperature advection is expressed as the product of the appropriate component of the wind – whether east-west (u) or north-south (v) – and the local change of temperature in some direction – east-west (x) or north-south (y) – where:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} \tag{14}$$

In vector notation, (14) can be written as:

$$advection = -\vec{\mathbf{v}} \cdot \nabla T \qquad (15)$$

The units of temperature advection are the units of wind – m s^{-1} – multiplied by the units of temperature – either °C or K – divided by distance units – m. As a result, temperature advection has units of °C s^{-1} or K s^{-1}; in other words, how temperature is changing *locally* over some finite amount of time Δt. We can evaluate (14) from charts of weather data using our centered finite difference approximation developed above.

Consider the hypothetical analysis presented in Figure 2. We are interested in computing the horizontal temperature advection at the point marked by the closed circle and wind observation. We have already completed an isotherm analysis using temperature data from this point as well as the other locations that surround it. We thus have everything we need to compute horizontal temperature advection.

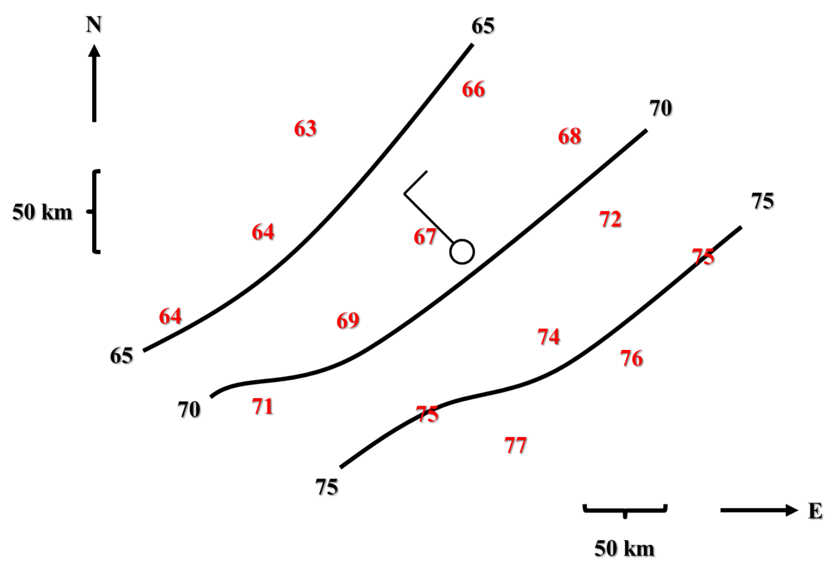

Figure 2. Hypothetical surface temperature observations (°F, red numbers), isotherm analysis (every 5°F, black lines), and a single wind observation (10 kt = 5.15 m s^{-1} out of the northwest). Depicted for reference are horizontal scales and the north and east cardinal directions. Data are plotted on a map constructed using the Mercator map projection.

To compute horizontal temperature advection, we must first set up our *x*- and *y*-axes. Fortunately, since we are told that the data are plotted on a Mercator map projection, the positive *x*-axis points to the right, or due east, while the positive *y*-axis points up, or due north. Since our centered finite difference approximation is only valid over finite distances – here, Δx and Δy – we must set up a small grid centered on the location of our wind observation. This is done so that we can estimate the temperature at points $x+1$, $x-1$, $y+1$, and $y-1$ – in other words, the terms that enter into the numerator of our centered finite differences. The result of doing so is given in Figure 3.

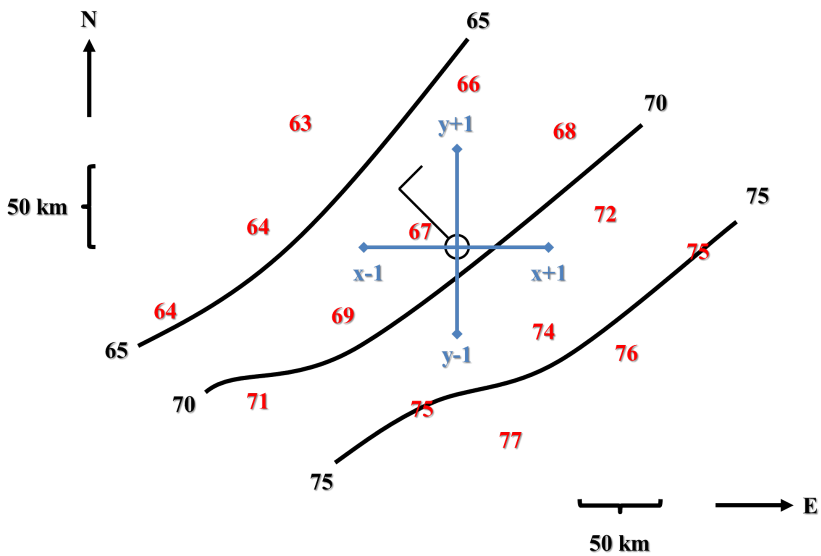

Figure 3. As in Figure 2, except with a finite grid drawn in centered on our wind observation. Both in this example and in practice, the distance Δx is taken to be equal to the distance Δy. In this case, using the distance references on the edges of the map, both Δx and Δy are 50 km (or 50,000 m).

Next, we use our isotherm analysis to estimate the value of temperature at points *x+1*, *x-1*, *y+1*, and *y-1*. We must do so because we do not have an exact temperature observation at any of these locations. Visually doing so, we state that the temperature at *x+1* is 72°F, at *x-1* is 67°F, at *y+1* is 67°F, and at *y-1* is 73°F. This enables us to compute the finite difference approximations to our partial derivatives, where:

$$advection = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} = -u\frac{T_{x+1} - T_{x-1}}{2\Delta x} - v\frac{T_{y+1} - T_{y-1}}{2\Delta y} = -u\frac{72°F - 67°F}{100000m} - v\frac{67°F - 73°F}{100000m} \qquad (16)$$

Note that, per Figure 3's caption, we know that $\Delta x = \Delta y = 50{,}000$ m, such that $2\Delta x = 2\Delta y = 100{,}000$ m. Now, we need to know the values of *u* and *v*, the zonal (east-west) and meridional (north-south) wind components, respectively. To obtain these values, we need to use a bit of trigonometry. Recall that in meteorological convention, from the north = 0°/360°, from the east = 90°, from the south = 180°, and from the west = 270°. If the wind direction (in degrees) is known, then the *u* and *v* components of the wind can be obtained using the following equations:

$$u = -\|\mathbf{v}\| \sin\left(\frac{\pi}{180} * wdir\right) \quad (17)$$

$$v = -\|\mathbf{v}\| \cos\left(\frac{\pi}{180} * wdir\right) \quad (18)$$

In both (17) and (18), ||**v**|| is the magnitude of the wind vector **v**. In applied terms, ||**v**|| is simply equal to the wind speed. The π/180 factor in both the sin and cos statements converts the wind direction from degrees to radians.

Returning to our example given by Figure 2, we know that the wind speed is equal to 10 kt = 5.15 m s^{-1}. We also know that our wind is out of (or from) the northwest. Expressed in degrees, from the northwest = 315° (e.g., halfway between 270°/west and 360°/north). If we substitute these values into (17) and (18), we obtain:

$$u = -5.15 ms^{-1} \sin\left(\frac{\pi}{180} * 315\right) = 3.64 ms^{-1} \quad (19)$$

$$v = -5.15 ms^{-1} \cos\left(\frac{\pi}{180} * 315\right) = -3.64 ms^{-1} \quad (20)$$

A bit of a sanity check is in order before proceeding. The positive x-axis is to the east, while the positive y-axis is to the north. Our wind is blowing ***from*** the north and west and, thus, ***to*** the south and east. Our wind thus blows in the positive x but negative y directions. Since u is along the x-axis (east-west) and v is along the y-axis (north-south), we would expect that u should be positive and v should be negative for a northwest wind – and, indeed, we find that this is true.

If we plug (19) and (20) into (16) and run through the calculations, we obtain:

$$advection = -3.64 ms^{-1} \frac{72°F - 67°F}{100000m} - \left(-3.64 ms^{-1}\right)\frac{67°F - 73°F}{100000m} = -0.0004°Fs^{-1} \quad (21)$$

In other words, due solely to horizontal advection, the temperature at the location of our wind observation is cooling by 0.0004°F every second. If we multiply this by 3,600 (the number of seconds in one hour) or 84,600 (the number of seconds in one day), we can convert this to °F h^{-1} or °F day^{-1}, respectively. Doing so, we obtain values of -1.44°F h^{-1} and -34.56°F day^{-1}. In other words, due solely to horizontal advection, the temperature at the location of our wind observation is cooling by 1.44°F every hour and 34.56°F every day.

Before we proceed further, it is again time for another sanity check. In Figure 2, we see that the wind is blowing toward the station from where it is colder. As a result, we would expect the wind to be advecting (or transporting) colder air toward the observation station. Our calculation suggests that this is true – due to advection, the temperature at the observation station is cooling.

The above calculation process represents a fairly complex means of evaluating horizontal temperature advection. By contrast, our sanity check hints at another, far less complex means of doing so. Instead of using Cartesian (x,y) coordinates, as we did before, we may use a *natural coordinate system* to assess horizontal temperature advection.

Recall that in the natural coordinate system, the appropriate coordinates become s, or along (**streamwise**) the wind, and n, or **normal** to the wind. For the example given in Figure 2, the positive s-axis points to the southeast, in the direction that the wind is blowing, and the positive n-axis points to the northeast, or 90° to the left of the positive s-axis. Figure 4 below provides a graphical depiction of the natural coordinate system applied to the example from Figures 2 and 3.

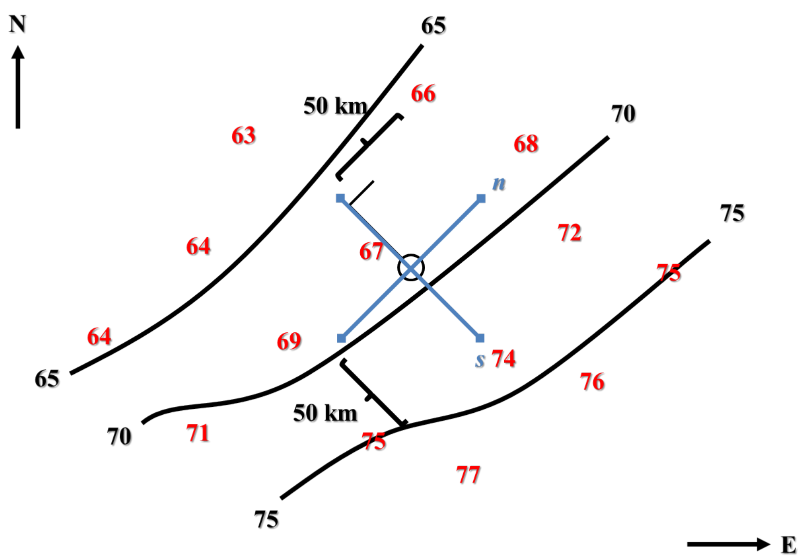

Figure 4. As in Figure 3, except with the finite grid drawn in the natural (rather than Cartesian) coordinate system. Both in this example and in practice, the distance Δs is taken to be equal to the distance Δn. In this case, using the distance references on the finite grid, both Δs and Δn are 50 km (or 50,000 m).

In the natural coordinate system, advection is expressed mathematically as:

$$advection = -\left\|\mathbf{v}\right\|\frac{\partial T}{\partial s} = -V\frac{\partial T}{\partial s} = -V\frac{T_{s+1} - T_{s-1}}{2\Delta s} \qquad (22)$$

In the above, V is the *wind speed*, equivalent to the magnitude of the velocity vector ($\|\mathbf{v}\|$). Here, we have used a second-order accurate centered finite difference approximation to obtain the relationship at the far end of Equation (22). Note that we no longer need to break down our wind into its u and v components, nor deal with a change in temperature in both the x and y directions. We only need to know the wind speed – 10 kt or 5.15 m s^{-1} in this example – and the change in temperature along the s-axis. Because the wind is aligned ***with*** the s-axis, the wind component perpendicular to the axis is zero, and thus we do not need to know the change in temperature along the n-axis.

Evaluating from Figure 4, we estimate that T_{s+1}, or the temperature at the grid point along the positive s-axis, is 73.5°F and that T_{s-1}, or the temperature at the grid point along the negative s-axis, is 66°F. Plugging these values into (22), we obtain:

$$advection = -\left(5.15ms^{-1}\right)\frac{73.5°F - 66°F}{2\left(50000m\right)} = -0.000386°Fs^{-1} \quad (23)$$

Note that this result is very nearly identical to that in Equation (21), as we expect using the same data. That this is true provides a sanity check upon our result. The two are not exactly equal to each other because of the inherent approximate nature to each of our two analyses, namely in obtaining the values of T at each of our grid points.

With the above in mind, we can state several general rules related to temperature advection:

- Where wind blows from *cold air toward warm air*, cold air advection is occurring. Where wind blows from *warm air toward cold air*, warm air advection is occurring.

- When the change in temperature over a fixed distance is large, the magnitude of the advection will be large. When the change in temperature over a fixed distance is small, the magnitude of the advection will be small.

- When the wind blows *parallel* to the isotherms, no horizontal temperature advection occurs. When the wind blows *perpendicular* to the isotherms, horizontal temperature advection is maximized.

- Horizontal temperature advection is larger when the wind component that blows perpendicular to the isotherms is larger. Horizontal temperature advection is smaller when the wind component that blows perpendicular to the isotherms is smaller.

- Horizontal temperature advection is one of *many* processes that can change temperature!

For Further Reading

Any college-level Calculus textbook will contain extensive information regarding the mathematical definition of limits, partial derivatives, and Taylor functions and series. Sections 1.2.2 and 1.2.3 of *Mid-Latitude Atmospheric Dynamics* by J. Martin provides similar information from the perspective of their application to the atmospheric sciences.