Numerical Methods for Solving First-Order Differential Equations

GENERAL REMARKS

A *numerical method* for solving an initial-value problem is a procedure that produces approximate solutions at particular points using only the operations of addition, subtraction, multiplication, division, and functional evaluations. In this chapter, we consider only first-order initial-value problems of the form

$$y' = f(x, y); \qquad y(x_0) = y_0 \tag{27.1}$$

Generalizations to higher-order problems are given in Chapter 28. Each numerical method will produce approximate solutions at the points $x_0, x_1, x_2, \ldots,$ where the difference between any two successive x-values is a constant step-size h; that is, $x_{n+1} - x_n = h$ $(n = 0, 1, 2, \ldots)$. Remarks made in Chapter 26 on the step-size remain valid for all the numerical methods presented below.

The approximate solution at x_n will be designated by $y(x_n)$, or simply y_n. The true solution at x_n will be denoted by either $Y(x_n)$ or Y_n. Note that once y_n is known, Eq. (*27.1*) can be used to obtain y'_n as

$$y'_n = f(x_n, y_n) \tag{27.2}$$

The simplest numerical method is Euler's method, described in Chapter 27.

A *predictor-corrector* method is a set of two equations for y_{n+1}. The first equation, called the *predictor,* is used to predict (obtain a first approximation to) y_{n+1}; the second equation, called the *corrector,* is then used to obtain a corrected value (second approximation) to y_{n+1}. In general, the corrector depends on the predicted value.

MODIFIED EULER'S METHOD

This is a simple predictor-corrector method that uses Euler's method (see Chapter 26) as the predictor and then uses the average value of y' at both the left and right end points of the interval $[x_n, x_{n+1}]$ $(n = 0, 1, 2, \ldots)$ as the slope of the line element approximation to the solution over that interval. The resulting equations are:

$$\text{predictor:} \qquad y_{n+1} = y_n + hy'_n$$

$$\text{corrector:} \qquad y_{n+1} = y_n + \frac{h}{2}(y'_{n+1} + y'_n)$$

For notational convenience, we designate the predicted value of y_{n+1} by py_{n+1}. It then follows from Eq. (*27.2*) that

$$py'_{n+1} = f(x_{n+1}, py_{n+1}) \tag{27.3}$$

The modified Euler's method becomes

$$\begin{aligned} &\text{predictor:} \qquad py_{n+1} = y_n + hy'_n \\ &\text{corrector:} \qquad y_{n+1} = y_n + \frac{h}{2}(py'_{n+1} + y'_n) \end{aligned} \tag{27.4}$$

RUNGE–KUTTA METHOD

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{27.5}$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h,\, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h,\, y_n + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_n + h,\, y_n + k_3)$$

This is *not* a predictor-corrector method.

ADAMS–BASHFORTH–MOULTON METHOD

$$\text{predictor:}\quad py_{n+1} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$$

$$\text{corrector:}\quad y_{n+1} = y_n + \frac{h}{24}(9py'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}) \tag{27.6}$$

MILNE'S METHOD

$$\text{predictor:}\quad py_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_n - y'_{n-1} + 2y'_{n-2})$$

$$\text{corrector:}\quad y_{n+1} = y_{n-1} + \frac{h}{3}(py'_{n+1} + 4y'_n + y'_{n-1}) \tag{27.7}$$

STARTING VALUES

The Adams–Bashforth–Moulton method and Milne's method require information at y_0, y_1, y_2, and y_3 to start. The first of these values is given by the initial condition in Eq. (*27.1*). The other three starting values are gotten by the Runge–Kutta method.

ORDER OF A NUMERICAL METHOD

A numerical method is of *order n*, where n is a positive integer, if the method is exact for polynomials of degree n or less. In other words, if the true solution of an initial-value problem is a polynomial of degree n or less, then the approximate solution and the true solution will be identical for a method of order n.

In general, the higher the order, the more accurate the method. Euler's method, Eq. (*26.4*), is of order one, the modified Euler's method, Eq. (*27.4*), is of order two, while the other three, Eqs. (*27.5*) through (*27.7*), are fourth-order methods.

Solved Problems

27.1. Use the modified Euler's method to solve $y' = y - x$; $y(0) = 2$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y - x$, $x_0 = 0$, and $y_0 = 2$. From Eq. (27.2) we have $y'_0 = f(0, 2) = 2 - 0 = 2$. Then using Eqs. (27.4) and (27.3), we compute

n = 0: $x_1 = 0.1$

$$py_1 = y_0 + hy'_0 = 2 + 0.1(2) = 2.2$$
$$py'_1 = f(x_1, py_1) = f(0.1, 2.2) = 2.2 - 0.1 = 2.1$$
$$y_1 = y_0 + \frac{h}{2}(py'_1 + y'_0) = 2 + 0.05(2.1 + 2) = 2.205$$
$$y'_1 = f(x_1, y_1) = f(0.1, 2.205) = 2.205 - 0.1 = 2.105$$

n = 1: $x_2 = 0.2$

$$py_2 = y_1 + hy'_1 = 2.205 + 0.1(2.105) = 2.4155$$
$$py'_2 = f(x_2, py_2) = f(0.2, 2.4155) = 2.4155 - 0.2 = 2.2155$$
$$y_2 = y_1 + \frac{h}{2}(py'_2 + y'_1) = 2.205 + 0.05(2.2155 + 2.105) = 2.421025$$
$$y'_2 = f(x_2, y_2) = f(0.2, 2.421025) = 2.421025 - 0.2 = 2.221025$$

n = 2: $x_3 = 0.3$

$$py_3 = y_2 + hy'_2 = 2.421025 + 0.1(2.221025) = 2.6431275$$
$$py'_3 = f(x_3, py_3) = f(0.3, 2.6431275) = 2.6431275 - 0.3 = 2.3431275$$
$$y_3 = y_2 + \frac{h}{2}(py'_3 + y'_2) = 2.421025 + 0.05(2.3431275 + 2.221025) = 2.6492326$$
$$y'_3 = f(x_3, y_3) = f(0.3, 2.6492326) = 2.6492326 - 0.3 = 2.3492326$$

Continuing in this manner, we generate Table 27-1. Compare it to Table 26-1.

Table 27-1

Method: MODIFIED EULER'S METHOD			
Problem: $y' = y - x;\ y(0) = 2$			
x_n	$h = 0.1$		True solution $Y(x) = e^x + x + 1$
	py_n	y_n	
0.0	—	2.0000000	2.0000000
0.1	2.2000000	2.2050000	2.2051709
0.2	2.4155000	2.4210250	2.4214028
0.3	2.6431275	2.6492326	2.6498588
0.4	2.8841559	2.8909021	2.8918247
0.5	3.1399923	3.1474468	3.1487213
0.6	3.4121914	3.4204287	3.4221188
0.7	3.7024715	3.7115737	3.7137527
0.8	4.0127311	4.0227889	4.0255409
0.9	4.3450678	4.3561818	4.3596031
1.0	4.7017999	4.7140808	4.7182818

27.2. Use the modified Euler's method to solve $y' = y^2 + 1$; $y(0) = 0$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y^2 + 1$, $x_0 = 0$, and $y_0 = 0$. From (*27.2*) we have $y_0' = f(0, 0) = (0)^2 + 1 = 1$. Then using (*27.4*) and (*27.3*), we compute

$n = 0$: $x_1 = 0.1$

$py_1 = y_0 + hy_0' = 0 + 0.1(1) = 0.1$

$py_1' = f(x_1, py_1) = f(0.1, 0.1) = (0.1)^2 + 1 = 1.01$

$y_1 = y_0 + (h/2)(py_1' + y_0') = 0 + 0.05(1.01 + 1) = 0.1005$

$y_1' = f(x_1, y_1) = f(0.1, 0.1005) = (0.1005)^2 + 1 = 1.0101003$

$n = 1$: $x_2 = 0.2$

$py_2 = y_1 + hy_1' = 0.1005 + 0.1(1.0101003) = 0.2015100$

$py_2' = f(x_2, py_2) = f(0.2, 0.2015100) = (0.2015100)^2 + 1 = 1.0406063$

$y_2 = y_1 + (h/2)(py_2' + y_1') = 0.1005 + 0.05(1.0406063) + 1.0101002) = 0.2030353$

$y_2' = f(x_2, y_2) = f(0.2, 0.2030353) = (0.2030353)^2 + 1 = 1.0412233$

$n = 2$: $x_3 = 0.3$

$py_3 = y_2 + hy_2' = 0.2030353 + 0.1(1.0412233) = 0.3071577$

$py_3' = f(x_3, py_3) = f(0.3, 0.3071577) = (0.3071577)^2 + 1 = 1.0943458$

$y_3 + y_2 + (h/2)(py_3' + y_2') = 0.2030353 + 0.05(1.0943458 + 1.0412233) = 0.3098138$

$y_3' = f(x_3, y_3) = f(0.3, 0.3098138) = (0.3098138)^2 + 1 = 1.0959846$

Continuing in this manner, we generate Table 27-2. Compare it to Table 26-3.

Table 27-2

Method: MODIFIED EULER'S METHOD			
Problem: $y' = y^2 + 1$; $y(0) = 0$			
x_n	$h = 0.1$		True solution $Y(x) = \tan x$
	py_n	y_n	
0.0	—	0.0000000	0.0000000
0.1	0.1000000	0.1005000	0.1003347
0.2	0.2015100	0.2030353	0.2027100
0.3	0.3071577	0.3098138	0.3093363
0.4	0.4194122	0.4234083	0.4227932
0.5	0.5413358	0.5470243	0.5463025
0.6	0.6769479	0.6848990	0.6841368
0.7	0.8318077	0.8429485	0.8422884
0.8	1.0140048	1.0298869	1.0296386
0.9	1.2359536	1·2592993	1.2601582
1.0	1.5178828	1.5537895	1.5574077

27.3. Find $y(1.6)$ for $y' = 2x$; $y(1) = 1$ using the modified Euler's method with $h = 0.2$.

Here $f(x, y) = 2x$, $x_0 = 1$, and $y_0 = 2$. From Eq. (*27.2*) we have $y_0' = f(1, 2) = 2(1) = 2$. Then using (*27.4*) and (*27.3*), we compute

$$
\begin{aligned}
\mathbf{n = 0:}\ \ & x_1 = x_0 + h = 1 + 0.2 = 1.2 \\
& py_1 = y_0 + hy_0' = 1 + 0.2(2) = 1.4 \\
& py_1' = f(x_1, py_1) = f(1.2, 1.4) = 2(1.2) = 2.4 \\
& y_1 = y_0 + (h/2)(py_1' + y_0') = 1 + 0.1(2.4 + 2) = 1.44 \\
& y_1' = f(x_1, y_1) = f(1.2, 1.44) = 2(1.2) = 2.4 \\
\mathbf{n = 1:}\ \ & x_2 = x_1 + h = 1.2 + 0.2 = 1.4 \\
& py_2 = y_1 + hy_1' = 1.44 + 0.2(2.4) = 1.92 \\
& py_2' = f(x_2, py_2) = f(1.4, 1.92) = 2(1.4) = 2.8 \\
& y_2 = y_1 + (h/2)(py_2' + y_1') = 1.44 + 0.1(2.8 + 2.4) = 1.96 \\
& y_2' = f(x_2, y_2) = f(1.4, 1.96) = 2(1.4) = 2.8 \\
\mathbf{n = 2:}\ \ & x_3 = x_2 + h = 1.4 + 0.2 = 1.6 \\
& py_3 = y_2 + hy_2' = 1.96 + 0.2(2.8) = 2.52 \\
& py_3' = f(x_3, py_3) = f(1.6, 2.52) = 2(1.6) = 3.2 \\
& y_3 = y_2 + (h/2)(py_3' + y_2') = 1.96 + 0.1(3.2 + 2.8) = 2.56
\end{aligned}
$$

The true solution is $Y(x) = x^2$; hence $Y(1.6) = y(1.6) = (1.6)^2 = 2.56$. Since the true solution is a second-degree polynomial and the modified Euler's method is a second-order method, this agreement is expected.

27.4. Use the Runge–Kutta method to solve $y' = y - x$; $y(0) = 2$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y - x$. Using Eq. (*27.5*) with $n = 0, 1, \ldots, 9$, we compute

$$
\begin{aligned}
\mathbf{n = 0:}\ \ & x_0 = 0, \qquad y_0 = 2 \\
& k_1 = hf(x_0, y_0) = hf(0, 2) = (0.1)(2 - 0) = 0.2 \\
& k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) = hf[0 + \tfrac{1}{2}(0.1), 2 + \tfrac{1}{2}(0.2)] \\
& \quad = hf(0.05, 2.1) = (0.1)(2.1 - 0.05) = 0.205 \\
& k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) = hf[0 + \tfrac{1}{2}(0.1), 2 + \tfrac{1}{2}(0.205)] \\
& \quad = hf(0.05, 2.103) = (0.1)(2.103 - 0.05) = 0.205 \\
& k_4 = hf(x_0 + h, y_0 + k_3) = hf(0 + 0.1, 2 + 0.205) \\
& \quad = hf(0.1, 2.205) = (0.1)(2.205 - 0.1) = 0.211 \\
& y_1 = y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
& \quad = 2 + \tfrac{1}{6}[0.2 + 2(0.205) + 2(0.205) + 0.211] = 2.205 \\
\mathbf{n = 1:}\ \ & x_1 = 0.1, \qquad y_1 = 2.205 \\
& k_1 = hf(x_1, y_1) = hf(0.1, 2.205) = (0.1)(2.205 - 0.1) = 0.211 \\
& k_2 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1) = hf[0.1 + \tfrac{1}{2}(0.1), 2.205 + \tfrac{1}{2}(0.211)] \\
& \quad = hf(0.15, 2.311) = (0.1)(2.311 - 0.15) = 0.216 \\
& k_3 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2) = hf[0.1 + \tfrac{1}{2}(0.1), 2.205 + \tfrac{1}{2}(0.216)]
\end{aligned}
$$

$$= hf(0.15, 2.313) = (0.1)(2.313 - 0.15) = 0.216$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = hf(0.1 + 0.1, 2.205 + 0.216)$$

$$= hf(0.2, 2.421) = (0.1)(2.421 - 0.2) = 0.222$$

$$y_2 = y_1 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.205 + \tfrac{1}{6}[0.211 + 2(0.216) + 2(0.216) + 0.222] = 2.421$$

n = 2: $x_2 = 0.2, \quad y_2 = 2.421$

$$k_1 = hf(x_2, y_2) = hf(0.2, 2.421) = (0.1)(2.421 - 0.2) = 0.222$$

$$k_2 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_1) = hf[0.2 + \tfrac{1}{2}(0.1), 2.421 + \tfrac{1}{2}(0.222)]$$

$$= hf(0.25, 2.532) = (0.1)(2.532 - 0.25) = 0.228$$

$$k_3 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_2) = hf[0.2 + \tfrac{1}{2}(0.1), 2.421 + \tfrac{1}{2}(0.228)]$$

$$= hf(0.25, 2.535) = (0.1)(2.535) - 0.25) = 0.229$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = hf(0.2 + 0.1, 2.421 + 0.229)$$

$$= hf(0.3, 2.650) = (0.1)(2.650 - 0.3) = 0.235$$

$$y_3 = y_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.421 + \tfrac{1}{6}[0.222 + 2(0.228) + 2(0.229) + 0.235] = 2.650$$

Continuing in this manner, we generate Table 27-3. Compare it with Table 27-1.

Table 27-3

Method: RUNGE–KUTTA METHOD		
Problem: $y' = y - x$; $y(0) = 2$		
x_n	$h = 0.1$ y_n	True solution $Y(x) = e^x + x + 1$
0.0	2.0000000	2.0000000
0.1	2.2051708	2.2051709
0.2	2.4214026	2.4214028
0.3	2.6498585	2.6498588
0.4	2.8918242	2.8918247
0.5	3.1487206	3.1487213
0.6	3.4221180	3.4221188
0.7	3.7137516	3.7137527
0.8	4.0255396	4.0255409
0.9	4.3596014	4.3596031
1.0	4.7182797	4.7182818

27.5. Use the Runge–Kutta method to solve $y' = y$; $y(0) = 1$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y$. Using Eq. (27.5) with $n = 0, 1, \ldots, 9$, we compute

n = 0: $x_0 = 0, \quad y_0 = 1$

$$k_1 = hf(x_0, y_0) = hf(0, 1) = (0.1)(1) = 0.1$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) = hf[0 + \tfrac{1}{2}(0.1), 1 + \tfrac{1}{2}(0.1)]$$

$$= hf(0.05, 1.05) = (0.1)(1.05) = 0.105$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) = hf[0 + \tfrac{1}{2}(0.1), 1 + \tfrac{1}{2}(0.105)]$$

$$= hf(0.05, 1.053) = (0.1)(1.053) = 0.105$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0 + 0.1, 1 + 0.105)$$

$$= hf(0.1, 1.105) = (0.1)(1.105) = 0.111$$

$$y_1 = y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \tfrac{1}{6}[0.1 + 2(0.105) + 2(0.105) + 0.111] = 1.105$$

$n = 1$: $x_1 = 0.1, \quad y_1 = 1.105$

$$k_1 = hf(x_1, y_1) = hf(0.1, 1.105) = (0.1)(1.105) = 0.111$$

$$k_2 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1) = hf[0.1 + \tfrac{1}{2}(0.1), 1.105 + \tfrac{1}{2}(0.111)]$$

$$= hf(0.15, 1.161) = (0.1)(1.161) = 0.116$$

$$k_3 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2) = hf[0.1 + \tfrac{1}{2}(0.1), 1.105 + \tfrac{1}{2}(0.116)]$$

$$= hf(0.15, 1.163) = (0.1)(1.163) = 0.116$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = hf(0.1 + 0.1, 1.105 + 0.116)$$

$$= hf(0.2, 1.221) = (0.1)(1.221) = 0.122$$

$$y_2 = y_1 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.105 + \tfrac{1}{6}[0.111 + 2(0.116) + 2(0.116) + 0.122] = 1.221$$

$n = 2$: $x_2 = 0.2, \quad y_2 = 1.221$

$$k_1 = hf(x_2, y_2) = hf(0.2, 1.221) = (0.1)(1.221) = 0.122$$

$$k_2 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_1) = hf[0.2 + \tfrac{1}{2}(0.1), 1.221 + \tfrac{1}{2}(0.122)]$$

$$= hf(0.25, 1.282) = (0.1)(1.282) = 0.128$$

$$k_3 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_2) = hf[0.2 + \tfrac{1}{2}(0.1), 1.221 + \tfrac{1}{2}(0.128)]$$

$$= hf(0.25, 1.285) = (0.1)(1.285) = 0.129$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = hf(0.2 + 0.1, 1.221 + 0.129)$$

$$= hf(0.3, 1.350) = (0.1)(1.350) = 0.135$$

$$y_3 = y_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.221 + \tfrac{1}{6}[0.122 + 2(0.128) + 2(0.129) + 0.135] = 1.350$$

Continuing in this manner, we generate Table 27-4.

27.6. Use the Runge–Kutta method to solve $y' = y^2 + 1$; $y(0) = 0$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y^2 + 1$. Using Eq. (*27.5*), we compute

$n = 0$: $x_0 = 0, \quad y_0 = 0$

$$k_1 = hf(x_0, y_0) = hf(0, 0) = (0.1)[(0)^2 + 1] = 0.1$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) + hf[0 + \tfrac{1}{2}(0.1), 0 + \tfrac{1}{2}(0.1)]$$

$$= hf(0.05, 0.05) = (0.1)[(0.05)^2 + 1] = 0.1$$

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) = hf[0 + \tfrac{1}{2}(0.1), 0 + \tfrac{1}{2}(0.1)]$$

$$= hf(0.05, 0.05) = (0.1)[(0.05)^2 + 1] = 0.1$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = hf[0 + 0.1, 0 + 0.1]$$

$$= hf(0.1, 0.1) = (0.1)[(0.1)^2 + 1] = 0.101$$

$$y_1 = y_0 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \tfrac{1}{6}[0.1 + 2(0.1) + 2(0.1) + 0.101] = 0.1$$

Table 27-4

Method: RUNGE–KUTTA METHOD		
Problem: $y' = y$; $y(0) = 1$		
x_n	$h = 0.1$ y_n	True solution $Y(x) = e^x$
0.0	1.0000000	1.0000000
0.1	1.1051708	1.1051709
0.2	1.2214026	1.2214028
0.3	1.3498585	1.3498588
0.4	1.4918242	1.4918247
0.5	1.6487206	1.6487213
0.6	1.8221180	1.8221188
0.7	2.0137516	2.0137527
0.8	2.2255396	2.2255409
0.9	2.4596014	2.4596031
1.0	2.7182797	2.7182818

$n = 1$: $x_1 = 0.1, \qquad y_1 = 0.1$

$$k_1 = hf(x_1, y_1) = hf(0.1, 0.1) = (0.1)[(0.1)^2 + 1] = 0.101$$
$$k_2 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1) = hf[0.1 + \tfrac{1}{2}(0.1), (0.1) + \tfrac{1}{2}(0.101)]$$
$$= hf(0.15, 0.151) = (0.1)[(0.151)^2 + 1] = 0.102$$
$$k_3 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2) = hf[0.1 + \tfrac{1}{2}(0.1), (0.1) + \tfrac{1}{2}(0.102)]$$
$$= hf(0.15, 0.151) = (0.1)[(0.151)^2 + 1] = 0.102$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = hf(0.1 + 0.1, 0.1 + 0.102)$$
$$= hf(0.2, 0.202) = (0.1)[(0.202)^2 + 1] = 0.104$$
$$y_2 = y_1 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0.1 + \tfrac{1}{6}[0.101 + 2(0.102) + 2(0.102) + 0.104] = 0.202$$

$n = 2$: $x_2 = 0.2, \qquad y_2 = 0.202$

$$k_1 = hf(x_2, y_2) = hf(0.2, 0.202) = (0.1)[(0.202)^2 + 1] = 0.104$$
$$k_2 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_1) = hf[0.2 + \tfrac{1}{2}(0.1), 0.202 + \tfrac{1}{2}(0.104)]$$
$$= hf(0.25, 0.254) = (0.1)[(0.254)^2 + 1] = 0.106$$
$$k_3 = hf(x_2 + \tfrac{1}{2}h, y_2 + \tfrac{1}{2}k_2) = hf[0.2 + \tfrac{1}{2}(0.1), 0.202 + \tfrac{1}{2}(0.106)]$$
$$= hf(0.25, 0.255) = (0.1)[(0.255)^2 + 1] = 0.107$$
$$k_4 = hf(x_2 + h, y_2 + k_3) = hf(0.2 + 0.1, 0.202 + 0.107)$$
$$= hf(0.3, 0.309) = (0.1)[(0.309)^2 + 1] = 0.110$$
$$y_3 = y_2 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0.202 + \tfrac{1}{6}[0.104 + 2(0.106) + 2(0.107) + 0.110] = 0.309$$

Continuing in this manner, we generate Table 27-5.

Table 27-5

Method:	RUNGE–KUTTA METHOD	
Problem:	$y' = y^2 + 1;\ y(0) = 0$	
x_n	$h = 0.1$ y_n	True solution $Y(x) = \tan x$
0.0	0.0000000	0.0000000
0.1	0.1003346	0.1003347
0.2	0.2027099	0.2027100
0.3	0.3093360	0.3093363
0.4	0.4227930	0.4227932
0.5	0.5463023	0.5463025
0.6	0.6841368	0.6841368
0.7	0.8422886	0.8422884
0.8	1.0296391	1.0296386
0.9	1.2601588	1.2601582
1.0	1.5574064	1.5574077

27.7. Use the Adams–Bashforth–Moulton method to solve $y' = y - x;\ y(0) = 2$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y - x$, $x_0 = 0$, and $y_0 = 2$. Using Table 27-3, we find the three additional starting values to be $y_1 = 2.2051708$, $y_2 = 2.4214026$, and $y_3 = 2.6498585$. Thus,

$$y_0' = y_0 - x_0 = 2 - 0 = 2 \qquad y_1' = y_1 - x_1 = 2.1051708$$

$$y_2' = y_2 - x_2 = 2.2214026 \qquad y_3' = y_3 - x_3 = 2.3498585$$

Then, using Eqs. (*27.6*), beginning with $n = 3$, and Eq. (*27.3*), we compute

$\mathbf{n = 3}$: $x_4 = 0.4$

$$\begin{aligned} py_4 &= y_3 + (h/24)(55y_3' - 59y_2' + 37y_1' - 9y_0') \\ &= 2.6498585 + (0.1/24)[55(2.349585) - 59(2.2214026) + 37(2.1051708) - 9(2)] \\ &= 2.8918201 \end{aligned}$$

$$py_4' = py_4 - x_4 = 2.8918201 - 0.4 = 2.4918201$$

$$\begin{aligned} y_4 &= y_3 + (h/24)(9py_4' + 19y_3' - 5y_2' + y_1') \\ &= 2.6498585 + (0.1/24)[9(2.4918201) + 19(2.3498585) - 5(2.2214026) + 2.1051708] \\ &= 2.8918245 \end{aligned}$$

$$y_4' = y_4 - x_4 = 2.8918245 - 0.4 = 2.4918245$$

$n = 4$: $x_5 = 0.5$

$$py_5 = y_4 + (h/24)(55y'_4 - 59y'_3 + 37y'_2 - 9y'_1)$$
$$= 2.8918245 + (0.1/24)[55(2.4918245) - 59(2.3498585) + 37(2.2214026) - 9(2.1051708)]$$
$$= 3.1487164$$
$$py'_5 = py_5 - x_5 = 3.1487164 - 0.5 = 2.6487164$$
$$y_5 = y_4 + (h/24)(9py'_5 + 19y'_4 - 5y'_3 + y'_2)$$
$$= 2.8918245 + (0.1/24)[9(2.6487164) + 19(2.4918245) - 5(2.3498585) + 2.2214026]$$
$$= 3.1487213$$
$$y'_5 = y_5 - x_5 = 3.1487213 - 0.5 = 2.6487213$$

$n = 5$: $x_6 = 0.6$

$$py_6 = y_5 + (h/24)(55y'_5 - 59y'_4 + 37y'_3 - 9y'_2)$$
$$= 3.1487213 + (0.1/24)[55(2.6487213) - 59(2.4918245) + 37(2.3498585) - 9(2.2214026)]$$
$$= 3.4221137$$
$$py'_6 = py_6 - x_6 = 3.4221137 - 0.6 = 2.8221137$$
$$y_6 = y_5 + (h/24)(9py'_6 + 19y'_5 - 5y'_4 + y'_3)$$
$$= 3.1487213 + (0.1/24)[9(2.8221137) + 19(2.6487213) - 5(2.4918245) + 2.3498585]$$
$$= 3.4221191$$
$$y'_6 = y_6 - x_6 = 3.4221191 - 0.6 = 2.8221191$$

Continuing in this manner, we generate Table 27-6.

Table 27-6

Method: ADAMS–BASHFORTH–MOULTON METHOD			
Problem: $y' = y - x;\ y(0) = 2$			
x_n	$h = 0.1$		True solution $Y(x) = e^x + x + 1$
	py_n	y_n	
0.0	—	2.0000000	2.0000000
0.1	—	2.2051708	2.2051709
0.2	—	2.4214026	2.4214028
0.3	—	2.6498585	2.6498588
0.4	2.8918201	2.8918245	2.8918247
0.5	3.1487164	3.1487213	3.1487213
0.6	3.4221137	3.4221191	3.4221188
0.7	3.7137473	3.7137533	3.7137527
0.8	4.0255352	4.0255418	4.0255409
0.9	4.3595971	4.3596044	4.3596031
1.0	4.7182756	4.7182836	4.7182818

27.8. Use the Adams–Bashforth–Moulton method to solve $y' = y^2 + 1$; $y(0) = 0$, on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y^2 + 1$, $x_0 = 0$, and $y_0 = 0$. Using Table 27-5, we find the three additional starting values to be $y_1 = 0.1003346$, $y_2 = 0.2027099$, and $y_3 = 0.3093360$. Thus,

$$y_0' = (y_0)^2 + 1 = (0)^2 + 1 = 1$$
$$y_1' = (y_1)^2 + 1 = (0.1003346)^2 + 1 = 1.0100670$$
$$y_2' = (y_2)^2 + 1 = (0.2027099)^2 + 1 = 1.0410913$$
$$y_3' = (y_3)^2 + 1 = (0.3093360)^2 + 1 = 1.0956888$$

Then, using Eqs. (*27.6*), beginning with $n = 3$, and Eq. (*27.3*), we compute

$n = 3$: $x_4 = 0.4$

$$\begin{aligned} py_4 &= y_3 + (h/24)(55y_3' - 59y_2' + 37y_1' - 9y_0') \\ &= 0.3093360 + (0.1/24)[55(1.0956888) - 59(1.0410913) + 37(1.0100670) - 9(1)] \\ &= 0.4227151 \end{aligned}$$

$$py_4' = (py_4)^2 + 1 = (0.4227151)^2 + 1 = 1.1786881$$

$$\begin{aligned} y_4 &= y_3 + (h/24)(9py_4' + 19y_3' - 5y_2' + y_1') \\ &= 0.3093360 + (0.1/24)[9(1.1786881) + 19(1.0956888) - 5(1.0410913) + 1.0100670] \\ &= 0.4227981 \end{aligned}$$

$$y_4' = (y_4)^2 + 1 = (0.4227981)^2 + 1 = 1.1787582$$

$n = 4$: $x_5 = 0.5$

$$\begin{aligned} py_5 &= y_4 + (h/24)(55y_4' - 59y_3' + 37y_2' - 9y_1') \\ &= 0.4227981 + (0.1/24)[55(1.1787582) - 59(1.0956888) + 37(1.0410913) - 9(1.0100670)] \\ &= 0.5461974 \end{aligned}$$

$$py_5' = (py_5)^2 + 1 = (0.5461974)^2 + 1 = 1.2983316$$

$$\begin{aligned} y_5 &= y_4 + (h/24)(9py_5' + 19y_4' - 5y_3' + y_2') \\ &= 0.4227981 + (0.1/24)[9(1.2983316) + 19(1.1787582) - 5(1.0956888) + 1.0410913] \\ &= 0.5463149 \end{aligned}$$

$$y_5' = (y_5)^2 + 1 = (0.5463149)^2 + 1 = 1.2984600$$

$n = 5$: $x_6 = 0.6$

$$\begin{aligned} py_6 &= y_5 + (h/24)(55y_5' - 59y_4' + 37y_3' - 9y_2') \\ &= 0.5463149 + (0.1/24)[55(1.2984600) - 59(1.1787582) + 37(1.0956888) - 9(1.0410913)] \\ &= 0.6839784 \end{aligned}$$

$$py_6' = (py_6)^2 + 1 = (0.6839784)^2 + 1 = 1.4678265$$

$$\begin{aligned} y_6 &= y_5 + (h/24)(9py_6' + 19y_5' - 5y_4' + y_3') \\ &= 0.5463149 + (0.1/24)[9(1.4678265) + 19(1.2984600) - 5(1.1787582) + 1.0956888] \\ &= 0.6841611 \end{aligned}$$

$$y_6' = (y_6)^2 + 1 = (0.6841611)^2 + 1 = 1.4680764$$

Continuing in this manner, we generate Table 27-7.

27.9. Use the Adams–Bashforth–Moulton method to solve $y' = 2xy/(x^2 - y^2)$; $y(1) = 3$ on the interval $[1, 2]$ with $h = 0.2$.

Here $f(x, y) = 2xy/(x^2 - y^2)$, $x_0 = 1$, and $y_0 = 3$. With $h = 0.2$, $x_1 = x_0 + h = 1.2$, $x_2 = x_1 + h = 1.4$, and $x_3 = x_2 + h = 1.6$. Using the Runge–Kutta method to obtain the corresponding y-values needed to

Table 27-7

Method: ADAMS–BASHFORTH–MOULTON METHOD			
Problem: $y' = y^2 + 1;\ y(0) = 0$			
x_n	$h = 0.1$		True solution $Y(x) = \tan x$
	py_n	y_n	
0.0	—	0.0000000	0.0000000
0.1	—	0.1003346	0.1003347
0.2	—	0.2027099	0.2027100
0.3	—	0.3093360	0.3093363
0.4	0.4227151	0.4227981	0.4227932
0.5	0.5461974	0.5463149	0.5463025
0.6	0.6839784	0.6841611	0.6841368
0.7	0.8420274	0.8423319	0.8422884
0.8	1.0291713	1.0297142	1.0296386
0.9	1.2592473	1.2602880	1.2601582
1.0	1.5554514	1.5576256	1.5574077

start the Adams–Bashforth–Moulton method, we find $y_1 = 2.8232844$, $y_2 = 2.5709342$, and $y_3 = 2.1321698$. It then follows from Eq. (*27.3*) that

$$y_0' = \frac{2x_0 y_0}{(x_0)^2 - (y_0)^2} = \frac{2(1)(3)}{(1)^2 - (3)^2} = -0.75$$

$$y_1' = \frac{2x_1 y_1}{(x_1)^2 - (y_1)^2} = \frac{2(1.2)(2.8232844)}{(1.2)^2 - (2.8232844)^2} = -1.0375058$$

$$y_2' = \frac{2x_2 y_2}{(x_2)^2 - (y_2)^2} = \frac{2(1.4)(2.5709342)}{(1.4)^2 - (2.5709342)^2} = -1.5481884$$

$$y_3' = \frac{2x_3 y_3}{(x_3)^2 - (y_3)^2} = \frac{2(1.6)(2.1321698)}{(1.6)^2 - (2.1321698)^2} = -3.4352644$$

Then, using Eqs. (*27.6*), beginning with $n = 3$, and Eq. (*27.3*), we compute

n = 3: $x_4 = 1.8$

$$\begin{aligned} py_4 &= y_3 + (h/24)(55y_3' - 59y_2' + 37y_1' - 9y_0') \\ &= 2.1321698 + (0.1/24)[55(-3.4352644) - 59(-1.5481884) + 37(-1.0375058) - 9(-0.75)] \\ &= 1.0552186 \end{aligned}$$

$$py_4' = \frac{2x_4 py_4}{(x_4)^2 - (py_4)^2} = \frac{2(1.8)(1.0552186)}{(1.8)^2 - (1.0552186)^2} = 1.7863919$$

$$\begin{aligned} y_4 &= y_3 + (h/24)(9py_4' + 19y_3' - 5y_2' + y_1') \\ &= 2.1321698 + (0.1/24)[9(1.7863919) + 19(-3.4352644) - 5(-1.5481884) + (-1.0375058)] \\ &= 1.7780943 \end{aligned}$$

$$y_4' = \frac{2x_4 y_4}{(x_4)^2 - (y_4)^2} = \frac{2(1.8)(1.7780943)}{(1.8)^2 - (1.7780943)^2} = 81.6671689$$

$n = 4$: $x_5 = 2.0$

$$\begin{aligned} py_5 &= y_4 + (h/24)(55y_4' - 59y_3' + 37y_2' - 9y_1') \\ &= 1.7780943 + (0.1/24)[55(81.6671689) - 59(-3.4352644) + 37(-1.5481884) - 9(-1.0375058)] \\ &= 40.4983398 \end{aligned}$$

$$py_5' = \frac{2x_5 py_5}{(x_5)^2 - (py_5)^2} = \frac{2(2.0)(40.4983398)}{(2.0)^2 - (40.4983398)^2} = -0.0990110$$

$$\begin{aligned} y_5 &= y_4 + (h/24)(9py_5' + 19y_4' - 5y_3' + y_2') \\ &= 1.7780943 + (0.1/24)[9(-0.0990110) + 19(81.6671689) - 5(-3.4352644) + (-1.5481884)] \\ &= 14.8315380 \end{aligned}$$

$$y_5' = \frac{2x_5 y_5}{(x_5)^2 - (y_5)^2} = \frac{2(2.0)(14.8315380)}{(2.0)^2 - (14.8315380)^2} = -0.2746905$$

These results are troubling because the corrected values are not close to the predicted values as they should be. Note that y_5 is significantly different from py_5 and y_4' is significantly different from py_4'. In any predictor-corrector method, the corrected values of y and y' represent a fine-tuning of the predicted values, and not a major change. When significant changes occur, they are often the result of numerical instability, which can be remedied by a smaller step-size. Sometimes, however, significant differences arise because of a singularity in the solution.

In the computations above, note that the derivative at $x = 1.8$, namely 81.667, generates a nearly vertical slope and suggests a possible singularity near 1.8. Figure 27-1 is a direction field for this differential equation. On this direction field we have plotted the points (x_0, y_0) through (x_4, y_4) as determined by the Adams–Bashforth–Moulton method and then sketched the solution curve through these points consistent with the direction field. The cusp between 1.6 and 1.8 is a clear indicator of a problem.

The analytic solution to the differential equation is given in Problem 3.14 as $x^2 + y^2 = ky$. Applying the initial condition, we find $k = 10/3$, and then using the quadratic formula to solve

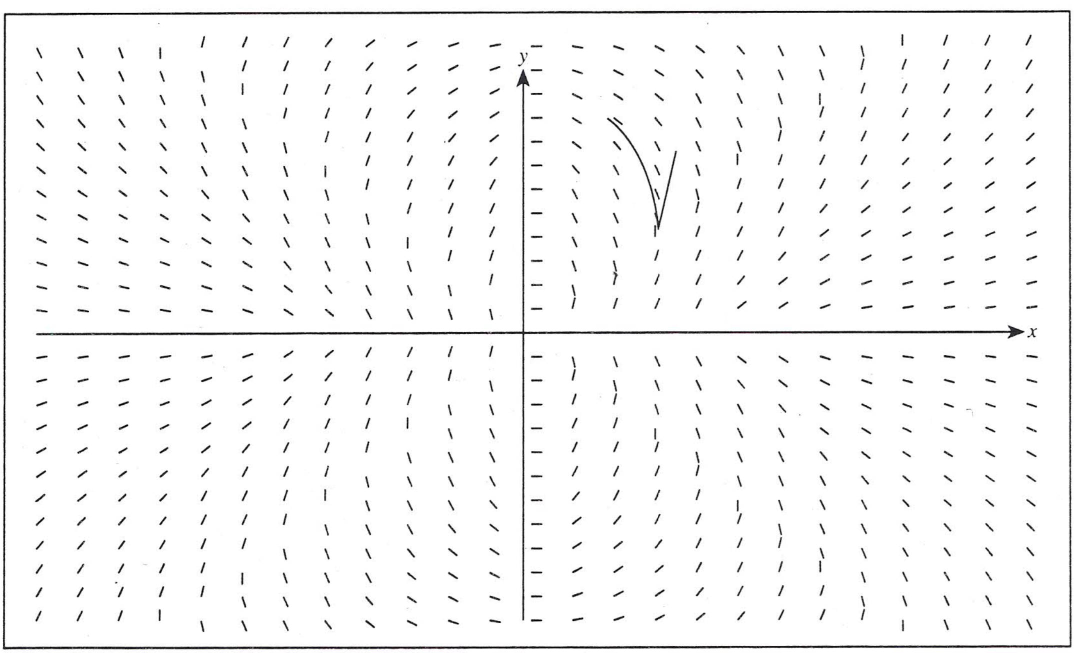

Fig. 27-1

explicitly for y, we obtain the solution

$$y = \frac{5 + \sqrt{25 - 9x^2}}{3}$$

This solution is only defined through $x = 5/3$ and is undefined after that.

27.10. Redo Problem 27.7 using Milne's method.

The values of y_0, y_1, y_2, y_3, and their derivatives are exactly as given in Problem 27.7. Using Eqs. (27.7) and (27.3), we compute

$$n = 3\text{:}\quad py_4 = y_0 + \frac{4h}{3}(2y_3' - y_2' + 2y_1')$$

$$= 2 + \frac{4(0.1)}{3}[2(2.3498585) - 2.2214026 + 2(2.1051708)]$$

$$= 2.8918208$$

$$py_4' = py_4 - x_4 = 2.4918208$$

$$y_4 = y_2 + \frac{h}{3}(py_4' + 4y_3' + y_2')$$

$$= 2.4214026 + \frac{0.1}{3}[2.4918208 + 4(2.3498585) + 2.2214026]$$

$$= 2.8918245$$

$$n = 4\text{:}\quad x_4 = 0.4, \qquad y_4' = y_4 - x_4 = 2.4918245$$

$$py_5 = y_1 + \frac{4h}{3}(2y_4' - y_3' + 2y_2')$$

$$= 2.2051708 + \frac{4(0.1)}{3}[2(2.4918245) - 2.3498585 + 2(2.2214026)]$$

$$= 3.1487169$$

$$py_5' = py_5 - x_5 = 2.6487169$$

$$y_5 = y_3 + \frac{h}{3}(py_5' + 4y_4' + y_3')$$

$$= 2.6498585 + \frac{0.1}{3}[2.6487169 + 4(2.4918245) + 2.3498585]$$

$$= 3.1487209$$

$$n = 5\text{:}\quad x_5 = 0.5, \qquad y_5' = y_5 - x_5 = 2.6487209$$

$$py_6 = y_2 + \frac{4h}{3}(2y_5' - y_4' + 2y_3')$$

$$= 2.4214026 + \frac{4(0.1)}{3}[2(2.6487209) - 2.4918245 + 2(2.3498585)]$$

$$= 3.4221138$$

$$py_6' = py_6 - x_6 = 2.8221138$$

$$y_6 = y_4 + \frac{h}{3}(py_6' + 4y_5' + y_4')$$

$$= 2.8918245 + \frac{0.1}{3}[2.8221138 + 4(2.6487209) + 2.4918245]$$

$$= 3.4221186$$

Continuing in this manner, we generate Table 27-8.

Table 27-8

Method: MILNE'S METHOD			
Problem: $y' = y - x;\ y(0) = 2$			
x_n	$h = 0.1$		True solution
	py_n	y_n	$Y(x) = e^x + x + 1$
0.0	—	2.0000000	2.0000000
0.1	—	2.2051708	2.2051709
0.2	—	2.4214026	2.4214028
0.3	—	2.6498585	2.6498588
0.4	2.8918208	2.8918245	2.8918247
0.5	3.1487169	3.1487209	3.1487213
0.6	3.4221138	3.4221186	3.4221188
0.7	3.7137472	3.7137524	3.7137527
0.8	4.0255349	4.0255407	4.0255409
0.9	4.3595964	4.3596027	4.3596031
1.0	4.7182745	4.7182815	4.7182818

27.11. Redo Problem 27.8 using Milne's method.

The values of y_0, y_1, y_2, y_3, and their derivatives are exactly as given in Problem 27.8. Using Eqs. *(27.7)* and *(27.3)*, we compute

$$\textbf{n = 3:}\quad py_4 = y_0 + \frac{4h}{3}(2y_3' - y_2' + 2y_1')$$

$$= 0 + \frac{4(0.1)}{3}[2(1.0956888) - 1.0410913 + 2(1.0100670)]$$

$$= 0.4227227$$

$$py_4' = (py_4)^2 + 1 = (0.4227227)^2 + 1 = 1.1786945$$

$$y_4 = y_2 + \frac{h}{3}(py_4' + 4y_3' + y_2')$$

$$= 0.2027099 + \frac{0.1}{3}[1.1786945 + 4(1.0956888) + 1.0410913]$$

$$= 0.4227946$$

$$\textbf{n = 4:}\quad x_4 = 0.4, \qquad y_4' = (y_4)^2 + 1 = (0.4227946)^2 + 1 = 1.1787553$$

$$py_5 = y_1 + \frac{4h}{3}(2y_4' - y_3' + 2y_2')$$

$$= 0.1003346 + \frac{4(0.1)}{3}[2(1.1787553) - 1.0956888 + 2(1.0410913)]$$

$$= 0.5462019$$

$$py_5' = (py_5)^2 + 1 = (0.5462019)^2 + 1 = 1.2983365$$

$$y_5 = y_3 + \frac{h}{3}(py_5' + 4y_4' + y_3')$$

$$= 0.3093360 + \frac{0.1}{3}[1.2983365 + 4(1.1787553) + 1.0956888]$$

$$= 0.5463042$$

$n = 5$: $x_5 = 0.5$, $\quad y_5' = (y_5)^2 + 1 = (0.5463042)^2 + 1 = 1.2984483$

$$py_6 = y_2 + \frac{4h}{3}(2y_5' - y_4' + 2y_3')$$

$$= 0.2027099 + \frac{4(0.1)}{3}[2(1.2984483) - 1.1787553 + 2(1.0956888)]$$

$$= 0.6839791$$

$$py_6' = (py_6)^2 + 1 = (0.6839791)^2 + 1 = 1.4678274$$

$$y_6 = y_4 + \frac{h}{3}(py_6' + 4y_5' + y_4')$$

$$= 0.4227946 + \frac{0.1}{3}[1.4678274 + 4(1.2984483) + 1.1787553]$$

$$= 0.6841405$$

Continuing in this manner, we generate Table 27-9.

Table 27-9

Method: MILNE'S METHOD			
Problem: $y' = y^2 + 1$; $y(0) = 0$			
x_n	$h = 0.1$		True solution $Y(x) = \tan x$
	py_n	y_n	
0.0	—	0.0000000	0.0000000
0.1	—	0.1003346	0.1003347
0.2	—	0.2027099	0.2027100
0.3	—	0.3093360	0.3093363
0.4	0.4227227	0.4227946	0.4227932
0.5	0.5462019	0.5463042	0.5463025
0.6	0.6839791	0.6841405	0.6841368
0.7	0.8420238	0.8422924	0.8422884
0.8	1.0291628	1.0296421	1.0296386
0.9	1.2592330	1.2601516	1.2601582
1.0	1.5554357	1.5573578	1.5574077

27.12. Use Milne's method to solve $y' = y$; $y(0) = 1$ on the interval $[0, 1]$ with $h = 0.1$.

Here $f(x, y) = y$, $x_0 = 0$, and $y_0 = 1$. From Table 27-4, we find as the three additional starting values $y_1 = 1.1051708$, $y_2 = 1.2214026$, and $y_3 = 1.3498585$. Note that $y_1' = y_1$, $y_2' = y_2$, and $y_3' = y_3$. Then, using Eqs. (*27.7*) and (*27.3*), we compute

$$\mathbf{n = 3:}\quad py_4 = y_0 + \frac{4h}{3}(2y_3' - y_2' + 2y_1')$$

$$= 1 + \frac{4(0.1)}{3}[2(1.3498585) - 1.2214026 + 2(1.1051708)]$$

$$= 1.4918208$$

$$py_4' = py_4 = 1.4918208$$

$$y_4 = y_2 + \frac{h}{3}(py_4' + 4y_3' + y_2')$$

$$= 1.2214026 + \frac{0.1}{3}[1.4918208 + 4(1.3498585) + 1.2214026]$$

$$= 1.4918245$$

$$\mathbf{n = 4:}\quad x_4 = 0.4, \qquad y_4' = y_4 = 1.4918245$$

$$py_5 = y_1 + \frac{4h}{3}(2y_4' - y_3' + 2y_2')$$

$$= 1.1051708 + \frac{4(0.1)}{3}[2(1.4918245) - 1.3498585 + 2(1.2214026)]$$

$$= 1.6487169$$

$$py_5' = py_5 = 1.6487169$$

$$y_5 = y_3 + \frac{h}{3}(py_5' + 4y_4' + y_3')$$

$$= 1.3498585 + \frac{0.1}{3}[1.6487169 + 4(1.4918245) + 1.3498585]$$

$$= 1.6487209$$

$$\mathbf{n = 5:}\quad x_5 = 0.5, \qquad y_5' = y_5 = 1.6487209$$

$$py_6 = y_2 + \frac{4h}{3}(2y_5' - y_4' + 2y_3')$$

$$= 1.2214026 + \frac{4(0.1)}{3}[2(1.6487209) - 1.4918245 + 2(1.3498585)]$$

$$= 1.8221138$$

$$py_6' = py_6 = 1.8221138$$

$$y_6 = y_4 + \frac{h}{3}(py_6' + 4y_5' + y_4')$$

$$= 1.4918245 + \frac{0.1}{3}[1.8221138 + 4(1.6487209) + 1.4918245]$$

$$= 1.8221186$$

Continuing in this manner, we generate Table 27-10.