FTCS

For $\frac{\partial \phi}{\partial t} = -u \frac{\partial \phi}{\partial x}$, write as

$$\frac{\phi_i^{\tau+1} - \phi_i^{\tau}}{\Delta t} = -U \frac{(\phi_{i+1}^{\tau} - \phi_{i-1}^{\tau})}{2\Delta x}$$

or ① $\phi_i^{\tau+1} = \phi_i^{\tau} - G\,[\phi_{i+1}^{\tau} - \phi_{i-1}^{\tau}]/2$; $G = \frac{U\Delta t}{\Delta x} \equiv$ Courant number

② Assume solution $\phi(x,t) = \hat{\phi}(k,\omega)\, e^{i(kx+\omega t)}$

where $x = n\Delta x$, $t = \tau\Delta t$, ω = frequency, k = wavenumber
$n = 0, \pm 1, \pm 2$, $\tau = 0, 1, 2$

so that the continous (x,t) space is replaced by a grid of discrete points.

Denote:

$$\phi_i^{\tau+1} = \phi_i^{\tau}\phi_0^{1} \;;\; \phi_{i+1}^{\tau} = \phi_i^{\tau}\phi_1^{0} \;;\; \phi_{i-1}^{\tau} = \phi_i^{\tau}\phi_{-1}^{0}$$

Substitute ② into ① and cancel common ϕ_i^{τ} factor, $\hat{\phi}$ factor

$$e^{i\omega\Delta t} = 1 - \frac{G}{2}\left[e^{ik\Delta x} - e^{-ik\Delta x}\right]$$

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$.

$$e^{i\omega\Delta t} = 1 - \frac{G}{2}\left[2i\sin(k\Delta x)\right] = 1 - G\left[i\sin(k\Delta x)\right]$$

Due to the imprecise approximation for the derivatives, ω is decomposed into real and imaginary parts

$\omega = \omega_r + i\omega_i$

Therefore

$$e^{i\omega\Delta t} = e^{i(\omega_r + i\omega_i)\Delta t} = e^{i\omega_r\Delta t} e^{-\omega_i\Delta t} = e^{i\omega_r\Delta t}\lambda$$

Where $\lambda = e^{-\omega_i \Delta t} \equiv$ amplitude change of the finite difference solution per time step.

¶ In the true solution, $\lambda = 1$, but in a finite difference scheme, often $\lambda \neq 1$.

To solve for λ, expand $e^{i\omega_r\Delta t}$ using Euler's formula

$$\lambda[\cos(\omega_r\Delta t) + i\sin(\omega_r\Delta t)] = 1 - q[i\sin(k\Delta x)]$$

The real and imaginary parts must separately be equal:

(3) $\lambda\cos(\omega_r\Delta t) = 1$

(4) $\lambda\sin(\omega_r\Delta t) = -q\sin(k\Delta x)$

$(3)^2 + (4)^2$ gives

$$\lambda^2 = 1 + q^2\sin^2(k\Delta x)$$

$$\lambda = \pm\sqrt{1 + q^2\sin^2(k\Delta x)}$$

Except for ~~small~~ very small k ($L \to \infty$), or for $\Delta x \to 0$, or for a $2\Delta x$ wave, $|\lambda| > 1$.

FTCS is linearly unstable!

FTBS (for $U_i > 0$) , i.e. forward in time, upstream

Will have

$$\phi_i^{\tau+1} = \phi_i^{\tau} - q\left(\phi_i^{\tau} - \phi_{i-1}^{\tau}\right);$$

Linear analysis gives

$$e^{i\omega\Delta t} = 1 - q\left(1 - e^{-ik\Delta x}\right)$$

or, applying λ and Euler's formula

$$\lambda\left(\cos(\omega_r\Delta t) + i\sin(\omega_r\Delta t)\right) = 1 - q + q\cos(k\Delta x) - qi\sin(k\Delta x)$$

Equating real and imaginary parts

(1) $\lambda\cos(\omega_r\Delta t) = 1 - q + q\cos(k\Delta x) = 1 - q(1 - \cos(k\Delta x))$

(2) $\lambda\sin(\omega_r\Delta t) = -q\sin(k\Delta x)$

or $\lambda^2 = \left[1 - q + q\cos(k\Delta x)\right]^2 + \left[q\sin(k\Delta x)\right]^2$

$$\lambda^2 = 1 + 2q(1 - \cos(k\Delta x)) + q^2(1 - 2\cos(k\Delta x) + \cos^2(k\Delta x)) + q^2\sin^2(k\Delta x)$$

$$\lambda^2 = 1 + 2q - 2q\cos(k\Delta x) + q^2 - 2q^2\cos(k\Delta x) + q^2$$

$$\boxed{\lambda^2 = 1 + 2q\left[1 - \cos(k\Delta x) + q^2 - 2q\cos(k\Delta x)\right]}$$

Turns out this is a damping scheme, with "mild" damping for small q ($q < 0.01$), strong damping for middle values of q (0.1 to 0.4) with very high at 0.5) preserved amplitude for $q = 1.0$, and amplification for $q > 1$. The damping is true for all $n\Delta x$, but is greatest at $n = 2$.

Phase speed

To get analytic phase speed c, ② / ①

$$\tan(\omega_r \Delta t) = ② / ①$$

Since $\omega = -ck = -c\frac{2\pi}{L_x} = -c\frac{2\pi}{n\Delta x}$

$$-c\frac{2\pi}{n\Delta x}\Delta t = \tan^{-1}\left(\frac{②}{①}\right)$$

The true phase speed is $C_{true} = U$, so to compare the analytic C and true C, divide by U

$$\frac{c}{U} = \frac{-n}{2\pi}\underbrace{\frac{\Delta x}{U\Delta t}}_{1/q}\tan^{-1}\left(\frac{②}{①}\right)$$

$$\boxed{\frac{C_{anal}}{U} = -\frac{n}{2\pi q}\tan^{-1}\left(\frac{②}{①}\right)}$$

$C_{anal}/U = 1.0$ at $q = 0.5, 1$. For $q < 0.5$, C_{anal} too slow, and it is worse for small n ($=0$ for $n=2$!). For $q > 0.5$, C_{anal} is too fast, and it is worse for small n.