Implicit scheme:

Consider

$$\frac{\phi_n^{\tau+1} - \phi_n^{\tau}}{\Delta t} = -\frac{U}{2}\left[\frac{\phi_{n+1}^{\tau+1} - \phi_{n-1}^{\tau+1}}{2\Delta X} + \frac{\phi_{n+1}^{\tau} - \phi_{n-1}^{\tau}}{2\Delta X}\right]$$

Must solve a system of simultaneous equations for $\phi_n^{\tau+1}$. Has no computational mode.

Turns out $|\lambda| = 1$, but has serious errors in phase velocity.

Euler Backward scheme (Matsuno scheme)

① $\phi_i^{*} = \phi_i^{\tau} - \frac{q}{2}\left[\phi_{i+1}^{\tau} - \phi_{i-1}^{\tau}\right]$

② $\phi_i^{\tau+1} = \phi_i^{\tau} - \frac{q}{2}\left[\phi_{i+1}^{*} - \phi_{i-1}^{*}\right]$

Rewrite ① as $\phi_i^{*} = \left[1 - \frac{q}{2}(\phi_1^{\circ} - \phi_{-1}^{\circ})\right]\phi_i^{\tau}$

$$\phi_i^{*} = \left[1 - \frac{q}{2}\left(e^{k\Delta x} - e^{-k\Delta x}\right)\right]\phi_i^{\tau}$$

Define $\sigma = q\sin(k\Delta x)$

$$\phi_i^{*} = \left[1 - \sigma i\right]\phi_i^{\tau}$$

Substitute ϕ_i^{*} into ②

$$\phi_i^{\tau+1} = \phi_i^{\tau} - \frac{q}{2}\left[1 - \sigma i\right]\left[\phi_{i+1}^{\tau} - \phi_{i-1}^{\tau}\right]$$

Assume solution $\phi = \hat{\phi} e^{i(k n \Delta x + \omega \tilde{n} \Delta t)}$

$$e^{i\omega \Delta t} = 1 - \frac{\phi}{2} [1 - \sigma i][2 i \sin(k \Delta x)]$$

$$e^{i\omega \Delta t} = 1 - [1 - \sigma i][i \sigma]$$

$$\lambda(\cos(\omega \Delta t) + i \sin(\omega \Delta t)) = 1 - \sigma^2 - \sigma i$$

(3) $\lambda \cos(\omega_r \Delta t) = 1 - \sigma^2$

(4) $\lambda \sin(\omega_r \Delta t) = -\sigma$

$(3)^2 + (4)^2$

$$\lambda^2 = 1 - 2\sigma^2 + \sigma^4 + \sigma^2$$

$$\boxed{\lambda^2 = 1 - \sigma^2 + \sigma^4} \qquad \therefore \boxed{\lambda < 1}$$

Some damping; max at $n=4$, $\phi = 0.707$

$$\boxed{\frac{c}{U} = \frac{-n}{2\pi \phi} \tan^{-1}\left(\frac{-\sigma}{1-\sigma^2}\right)}$$

Some phase lag for low ϕ, phase acceleration for high ϕ, except for low n. Low n has strongest damping and phase lag.