Leap frog - CTCS

$$\phi_i^{\tau+1} = \phi_i^{\tau-1} - q\left(\phi_{i+1}^{\tau} - \phi_{i-1}^{\tau}\right)$$

$$e^{i\omega\Delta t} = e^{-i\omega\Delta t} - q\underbrace{\left[e^{ik\Delta x} - e^{-ik\Delta x}\right]}_{2i\sin(k\Delta x)}$$

Denote $\sigma = q\sin(k\Delta x)$

Multiply by $e^{i\omega\Delta t}$

$$e^{2i\omega\Delta t} = 1 - 2i\sigma e^{i\omega\Delta t}$$

Denote $\Psi = e^{i\omega\Delta t}$

$$\Psi^2 + 2i\sigma\Psi - 1 = 0$$

$$\Psi = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, \quad b = 2i\sigma, \quad c = -1$$

$$\Psi = -i\sigma \pm \frac{1}{2}\sqrt{-4\sigma^2 + 4}$$

$$\Psi = -i\sigma \pm \sqrt{1-\sigma^2}$$

Now write as

$$\lambda\left(\cos(\omega_r\Delta t) + i\sin(\omega_r\Delta t)\right) = -i\sigma \pm \sqrt{1-\sigma^2}$$

Solving for λ in terms of real and imaginary parts

$$\text{(1)}\quad \lambda \cos(\omega_r \Delta t) = \pm\sqrt{1-\sigma^2} \qquad \text{if } \sigma^2 \le 1$$
$$\text{(2)}\quad \lambda \sin(\omega_r \Delta t) = -\sigma$$

$$\text{(3)}\quad \lambda \cos(\omega_r \Delta t) = 0 \qquad \text{if } \sigma^2 > 1$$
$$\text{(4)}\quad \lambda \sin(\omega_r \Delta t) = \sigma \pm \sqrt{\sigma^2-1}$$

When $\sigma^2 \le 1$, $(1)^2 + (2)^2$

$$\lambda^2 = 1-\sigma^2 + \sigma^2 = 1$$

Amplitude is preserved when $|q \sin(k\Delta x)| \le 1$,
or since $\sin^2(k\Delta x)$ is a maximum for $n=4$, $|q| \le 1$.

For $\sigma^2 > 1$ $\quad (3)^2 + (4)^2$

For 2n
$|q| \le \frac{1}{2}$

For 3n, $|q| \le \frac{1}{\sqrt{3}}$

$$\lambda^2 = \sigma^2 \pm 2\sqrt{\sigma^2-1} + \sigma^2 - 1$$
$$\lambda^2 = 2\sigma^2 \pm 2\sqrt{\sigma^2-1} - 1$$

Denote $\sigma^2 = (1+\varepsilon)^2 = 1 + 2\varepsilon + \varepsilon^2$

$$\lambda^2 = 2 + 4\varepsilon + 2\varepsilon^2 \pm 2\sqrt{(2\varepsilon+\varepsilon^2)} - 1$$
$$= 1 + 4\varepsilon + 2\varepsilon^2 \pm \sqrt{8\varepsilon + 4\varepsilon^2}$$

Hence $\lambda^2 > 1$. If $\sigma^2 > 1$, leapfrog is unstable!

Is $c > 1$ possible for $n > 4$? No. The round-off on computers will eventually introduce all Fourier components. $|q| \le 1$ always!

How about phase speeds? Since $\lambda = 1$ use ② or ①

$$\omega_r \Delta t = \sin^{-1}(-q \sin(k\Delta x))$$

$$-C_a \frac{2\pi \Delta t}{n\Delta x} = \sin^{-1}(-q \sin(k\Delta x))$$

$$\frac{C_{anal}}{U} = -\frac{n}{2\pi q} \sin^{-1}(-q \sin(k\Delta x))$$

Because leapfrog is quadratic, will have two wave solutions (picks up quadrants 1 and 3 in $\sin^{-1}$). There is physical mode and a computational mode. The computational mode occurs due to using a 2nd order equation to approximate a 1st order differential equation. It travels opposite of C_{true}, and oscillates per timestep.

To help keep the odd and even time steps from decoupling, often an "Asselin filter" is used

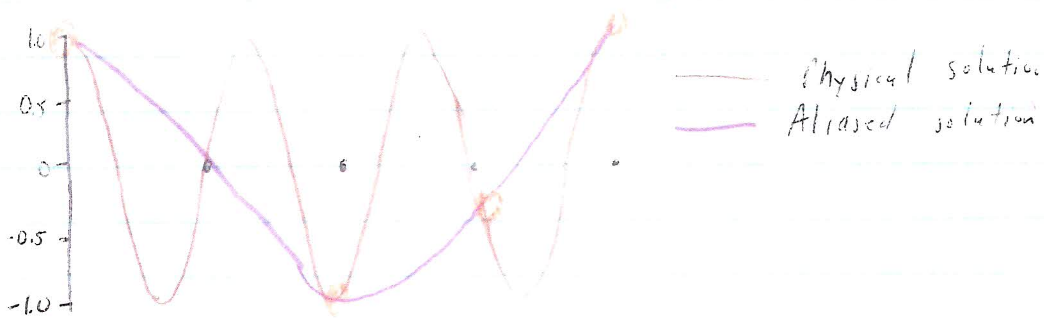
$$\bar{\phi}_i^{\tau-1} = \phi_i^{\tau-1} + \alpha\,(\phi_i^{\tau} + \bar{\phi}_i^{\tau-2} - 2\,\phi_i^{\tau-1})$$

where $\alpha = 0.25$. Problem is it aggravates the phase lag problem, and restricts the Courant stability problem.

Another problem: for the physical mode, $C_{anal}/C_{true} < 1$ except when $q = 1$, and this phase lag increases as q decreases and as n decreases. For $n = 2$, wave does not move!

Hence, leapfrog is called a "dispersive" scheme. This can cause further computational problems even though $\lambda = 1$, due to fictitious dispersion of waves. One error is aliasing (nonlinear instability).

Aliasing - when a wave smaller than $2\Delta x$ is fictitiously represented as a larger wave. In a long-term model, you can have a fictitious build-up of KE and it will "blow-up."



Solution to aliasing

1) Try to parameterize subgrid scale correlations so that energy is extracted in a manner consistent with reality
2) Use a spatial filter which removes shortest waves but leaves the longer ones relatively unaffected.
3) Conserve enstrophy
4) Spectral methods or sometimes pseudo-spectral methods

---

1) $\overline{u_i' u_j'} = K_H \frac{\partial \bar{u}_i}{\partial x_j} \quad (i=1,2)$

$$K_H = K_{H_0} + k_0^2 (\Delta x)^2 |Def| \quad ; \quad k_0 \equiv \text{Von Karman constant} = 0.4$$

$$|Def| = \left[\frac{1}{2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right]^{\frac{1}{2}}$$

in cyclindrical coordinates, $|Def| = r \left(\left[\frac{\partial}{\partial r} \left(\frac{u_r}{r} \right) \right]^2 + \left[\frac{\partial}{\partial r} \left(\frac{v}{r} \right) \right]^2 \right)^{\frac{1}{2}}$

and

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} K_H \frac{\partial u}{\partial x}$$

2) A filter which looks like

$$\overline{\phi}_i^{\tau+1} = \phi_i^{\tau+1} + \beta\left(\phi_{i+1}^{\tau+1} + \phi_{i-1}^{\tau+1} - 2\phi_i^{\tau+1}\right)$$

$\beta \approx \frac{1}{16}$ on avg

$\hat{E}$ grid!

3) $\frac{\partial u}{\partial t} = -J_A(\overline{\psi}^{x}, u)$; this supposedly controls nonlinear energy cascade.

ψ = streamfunction?

$u = -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}$

$$J_A = \frac{1}{3}\left(J^{++} + J^{+x} + J^{x+}\right)$$

which conserved avg vorticity, enstrophy, KE and prevents unlimited energy growth due to aliasing

$$J_A = (\psi, u) = \underbrace{\frac{\partial \psi}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial u}{\partial x}}_{J^{++}} = \underbrace{\frac{\partial}{\partial x}\left(\psi\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi\frac{\partial u}{\partial x}\right)}_{J^{+x}}$$

$$= \underbrace{\frac{\partial}{\partial y}\left(u\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial \psi}{\partial y}\right)}_{J^{x+}}$$

on grid