Name____________________________

Numerical methods
Differential equations and modeling exercises

1) Use Heun's 2nd order method:

$$\psi_{n+1} = \psi_n + \frac{1}{2}\Delta t(k_1 + k_2); \qquad k_1 = f(t_n, \psi_n); \quad k_2 = f(t_n + \Delta t, \psi_n + k_1 \Delta t)$$

to solve:

$$\frac{d\psi}{dt} = t - \psi; \qquad \psi = 2 \quad \text{at} \quad t = 0$$

at t=0.3. Use the following table for guidance and $\Delta t = 0.1$. Also compute the analytical solution $\psi = 3e^{-t} + t - 1$ in the last column. Don't fill anything in the yellow section, the initial condition for ψ has already been provided.

t	n	k_1	adjusted t_2	adjusted ψ_2	k_2	ψ_{n+1}	$\psi_{true} = 3e^{-t} + t - 1$
0.0	0					2	2
0.1	1						
0.2	2						
0.3	3						

2) Use 4th-order traditional Runge-Kutta method

$$\psi_{n+1} = \psi_n + \frac{1}{6}\Delta t(k_1 + 2k_2 + 2k_3 + k_4); \qquad k_1 = f(t_n, \psi_n); \quad k_2 = f(t_n + \frac{1}{2}\Delta t, \psi_n + \frac{1}{2}k_1\Delta t) ;$$

$$k_3 = f(t_n + \frac{1}{2}\Delta t, \psi_n + \frac{1}{2}k_2\Delta t) \; ; \qquad k_4 = f(t_n + \Delta t, \psi_n + k_3\Delta t)$$

For the same problem in question 1 out to t=0.1 (just one time step).

t	n	k_1	adjusted t_2	adjusted ψ_2	k_2	adjusted t_3	adjusted ψ_3
0.0	0						
0.1	1						

t	n	k_3	adjusted t_4	adjusted ψ_4	k_4	ψ_{n+1}	ψ_{true}
0.0	0					2	2
0.1	1						

3) The temperature advection equation $\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}$ is written in the leapfrog scheme as:

$$\frac{T_i^{\tau+1} - T_i^{\tau-1}}{2\Delta t} = -u\frac{T_{i+1}^{\tau} - T_{i-1}^{\tau}}{2\Delta x}$$

Solve for $T_i^{\tau+1}$

4) Suppose one side of room is temporarily hotter than the other. The diffusion transfer of temperature to is represented by the 1D heat conductivity equation $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$ where κ is the thermal diffusivity.

Suppose temperature is represented by n=1 and n=2 wave solutions [T(x,t)= $T(x, t, n = 1) + T(x, t, n = 2)$] such that:

$$T(x, t, n = 1) = A_1(t) \sin\left(\frac{2\pi x}{L}\right) \quad ; \quad T(x, t, n = 2) = A_2(t) \sin\left(\frac{4\pi x}{L}\right)$$

Substitute these expressions into $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$, and write solutions for $\frac{\partial A_1}{\partial t}$ and $\frac{\partial A_2}{\partial t}$. Remember that the derivative of $\sin cx$ is $c \cos cx$, and the derivative of $c \cos cx$ is $-c^2 \sin cx$, where c is a constant.

What are 3 advantages of such spectral methods?

5) List the five steps to perform a numerical forecast

6) Define parameterization, and list three physical processes which much be parameterized

7) A buoy observes a wind of $\overrightarrow{|V|}$ = 15 ms^{-1}. We'll designate this as $\overrightarrow{|V|}_{ob}$. The previous model forecast give a background wind of 12 ms^{-1}. We'll designate this as $\overrightarrow{|V|}_{back}$. Suppose the standard deviation of the buoy observation error σ_{ob} =1 ms^{-1}, and the model background error σ_{back} =2 ms^{-1}. Find the analysis wind speed using:

$$|\vec{V}|_a = |\vec{V}|_{back} + W\left(|\vec{V}|_{ob} - |\vec{V}|_{back}\right); \qquad W = \frac{\sigma_{back}^2}{\sigma_{back}^2+\sigma_{ob}^2}$$

This is the process of data assimilation. Define data assimilation.