3.2 The Hydrostatic Equation

Air pressure at any height in the atmosphere is due to the force per unit area exerted by the weight of all of the air lying above that height. Consequently, atmospheric pressure decreases with increasing height above the ground (in the same way that the pressure at any level in a stack of foam mattresses depends on how many mattresses lie above that level). The net upward force acting on a thin horizontal slab of air, due to the decrease in atmospheric pressure with height, is generally very closely in balance with the downward force due to gravitational attraction that acts on the slab. If the net upward force on the slab is equal to the downward force on the slab, the atmosphere is said to be in *hydrostatic balance*. We will now derive an important equation for the atmosphere in hydrostatic balance.

Consider a vertical column of air with unit horizontal cross-sectional area (Fig. 3.1). The mass of air between heights z and $z + \delta z$ in the column is $\rho \delta z$, where ρ is the density of the air at height z. The downward force acting on this slab of air due to the weight of the air is $g\rho\delta z$, where g is the acceleration due to gravity at height z. Now let us consider the net

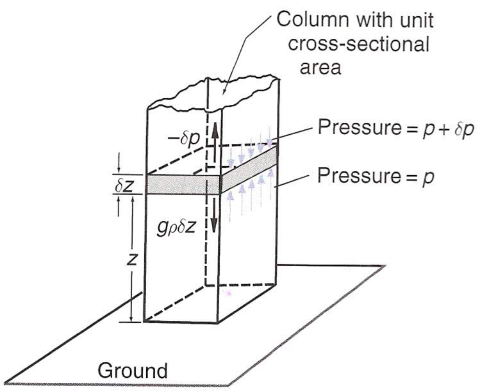

Fig. 3.1 Balance of vertical forces in an atmosphere in which there are no vertical accelerations (i.e., an atmosphere in hydrostatic balance). Small blue arrows indicate the downward force exerted on the air in the shaded slab due to the pressure of the air above the slab; longer blue arrows indicate the upward force exerted on the shaded slab due to the pressure of the air below the slab. Because the slab has a unit cross-sectional area, these two pressures have the same numerical values as forces. The net upward force due to these pressures ($-\delta p$) is indicated by the upward-pointing thick black arrow. Because the incremental pressure change δp is a negative quantity, $-\delta p$ is positive. The downward-pointing thick black arrow is the force acting on the shaded slab due to the mass of the air in this slab.

vertical force that acts on the slab of air between z and $z + \delta z$ due to the pressure of the surrounding air. Let the change in pressure in going from height z to height $z + \delta z$ be δp, as indicated in Fig. 3.1. Because we know that pressure decreases with height, δp must be a negative quantity, and the upward pressure on the lower face of the shaded block must be slightly greater than the downward pressure on the upper face of the block. Therefore, the net vertical force on the block due to the vertical gradient of pressure is upward and given by the positive quantity $-\delta p$, as indicated in Fig. 3.1. For an atmosphere in hydrostatic balance, the balance of forces in the vertical requires that

$$-\delta p = g\rho\delta z$$

or, in the limit as $\delta z \to 0$,

$$\frac{\partial p}{\partial z} = -g\rho \tag{3.17}$$

Equation (3.17) is the *hydrostatic equation*.[14] It should be noted that the negative sign in (3.17) ensures that the pressure decreases with increasing height. Because $\rho = 1/\alpha$ (3.17) can be rearranged to give

$$g dz = -\alpha dp \tag{3.18}$$

If the pressure at height z is $p(z)$, we have, from (3.17), above a fixed point on the Earth

$$-\int_{p(z)}^{p(\infty)} dp = \int_z^{\infty} g\rho dz$$

or, because $p(\infty) = 0$,

$$p(z) = \int_z^{\infty} g\rho dz \tag{3.19}$$

That is, the pressure at height z is equal to the weight of the air in the vertical column of unit cross-sectional area lying above that level. If the mass of the Earth's atmosphere were distributed uniformly over the globe, retaining the Earth's topography in its present form, the pressure at sea level would be 1.013×10^5 Pa, or 1013 hPa, which is referred to as *1 atmosphere* (or *1 atm*).

3.2.1 Geopotential

The *geopotential* Φ at any point in the Earth's atmosphere is defined as the work that must be done against the Earth's gravitational field to raise a mass of 1 kg from sea level to that point. In other words, Φ is the gravitational potential per unit mass. The units of geopotential are J kg^{-1} or m^2 s^{-2}. The force (in newtons) acting on 1 kg at height z above sea level is numerically equal to g. The work (in joules) in raising 1 kg from z to $z + dz$ is $g dz$; therefore

$$d\Phi \equiv g dz$$

or, using (3.18),

$$d\Phi \equiv g dz = -\alpha dp \tag{3.20}$$

[14] In accordance with Eq. (1.3), the left-hand side of (3.17) is written in partial differential notation, i.e., $\partial p/\partial z$, because the variation of pressure with height is taken with other independent variables held constant.

The geopotential $\Phi(z)$ at height z is thus given by

$$\Phi(z) = \int_0^z g dz \tag{3.21}$$

where the geopotential $\Phi(0)$ at sea level ($z = 0$) has, by convention, been taken as zero. The geopotential at a particular point in the atmosphere depends only on the height of that point and not on the path through which the unit mass is taken in reaching that point. The work done in taking a mass of 1 kg from point A with geopotential Φ_A to point B with geopotential Φ_B is $\Phi_B - \Phi_A$.

We can also define a quantity called the *geopotential height* Z as

$$Z \equiv \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g dz \tag{3.22}$$

where g_0 is the globally averaged acceleration due to gravity at the Earth's surface (taken as 9.81 m s^{-2}). Geopotential height is used as the vertical coordinate in most atmospheric applications in which energy plays an important role (e.g., in large-scale atmospheric motions). It can be seen from Table 3.1 that the values of z and Z are almost the same in the lower atmosphere where $g_0 \simeq g$.

In meteorological practice it is not convenient to deal with the density of a gas, ρ, the value of which is generally not measured. By making use of (3.2) or (3.15) to eliminate ρ in (3.17), we obtain

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}$$

Rearranging the last expression and using (3.20) yields

$$d\Phi = g\,dz = -RT\frac{dp}{p} = -R_d T_v \frac{dp}{p} \tag{3.23}$$

Table 3.1 Values of geopotential height (Z) and acceleration due to gravity (g) at 40° latitude for geometric height (z)

z (km)	Z (km)	g (m s^{-2})
0	0	9.81
1	1.00	9.80
10	9.99	9.77
100	98.47	9.50
500	463.6	8.43

If we now integrate between pressure levels p_1 and p_2, with geopotentials Φ_1 and Φ_2, respectively,

$$\int_{\Phi_1}^{\Phi_2} d\Phi = -\int_{p_1}^{p_2} R_d T_v \frac{dp}{p}$$

or

$$\Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p}$$

Dividing both sides of the last equation by g_0 and reversing the limits of integration yields

$$Z_2 - Z_1 = \frac{R_d}{g_0} \int_{p_2}^{p_1} T_v \frac{dp}{p} \qquad (3.24)$$

This difference $Z_2 - Z_1$ is referred to as the (geopotential) *thickness* of the layer between pressure levels p_1 and p_2.

The temperature of the atmosphere generally varies with height and the virtual temeprature correction cannot always be neglected. In this more general case (3.24) may be integrated if we define a mean virtual temperature $\overline{T}_v$ with respect to p as shown in Fig. 3.2. That is,

$$\overline{T}_v \equiv \frac{\int_{p_2}^{p_1} T_v \, d(\ln p)}{\int_{p_2}^{p_1} d(\ln p)} = \frac{\int_{p_2}^{p_1} T_v \frac{dp}{p}}{\ln\left(\frac{p_1}{p_2}\right)} \tag{3.28}$$

Then, from (3.24) and (3.28),

$$Z_2 - Z_1 = \overline{H} \ln\left(\frac{p_1}{p_2}\right) = \frac{R_d \overline{T}_v}{g_0} \ln\left(\frac{p_1}{p_2}\right) \tag{3.29}$$

Equation (3.29) is called the *hypsometric equation.*

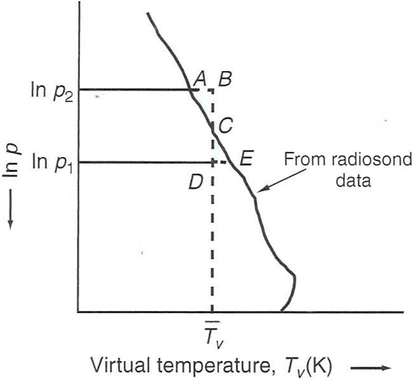

Fig. 3.2 Vertical profile, or sounding, of virtual temperature. If area ABC = area CDE, $\overline{T}_v$ is the mean virtual temperature with respect to ln p between the pressure levels p_1 and p_2.

3.2.3 Thickness and Heights of Constant Pressure Surfaces

Because pressure decreases monotonically with height, pressure surfaces (i.e., imaginary surfaces on which pressure is constant) never intersect. It can be seen from (3.29) that the thickness of the layer between any two pressure surfaces p_2 and p_1 is proportional to the mean virtual temperature of the layer, $\overline{T}_v$. We can visualize that as $\overline{T}_v$ increases, the air between the two pressure levels expands and the layer becomes thicker.

Exercise 3.4 Calculate the thickness of the layer between the 1000- and 500-hPa pressure surfaces (a) at a point in the tropics where the mean virtual temperature of the layer is 15 °C and (b) at a point in the polar regions where the corresponding mean virtual temperature is −40 °C.

Solution: From (3.29)

$$\Delta Z = Z_{500\ \text{hPa}} - Z_{1000\ \text{hPa}} = \frac{R_d \overline{T}_v}{g_0} \ln\left(\frac{1000}{500}\right) = 20.3\,\overline{T}_v\ \text{m}$$

Therefore, for the tropics with $\overline{T}_v = 288$ K, $\Delta Z = 5846$ m. For polar regions with $\overline{T}_v = 233$ K, $\Delta Z = 4730$ m. In operational practice, thickness is rounded to the nearest 10 m and is expressed in decameters (dam). Hence, answers for this exercise would normally be expressed as 585 and 473 dam, respectively. ■