Curvature and vectors

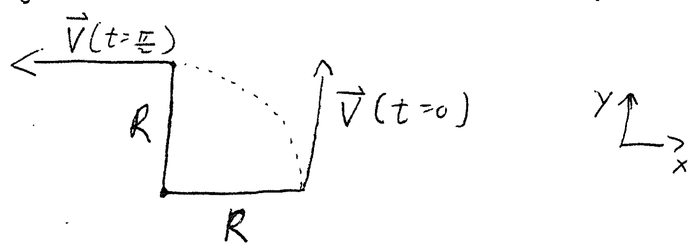
Consider the position vector $\vec{r}(x,y,t)$ along a circle with a radius of curvature R:

$$(1) \quad \vec{r} = R\cos t\,\hat{i} + R\sin t\,\hat{j}$$

The velocity vector is then

$$(2) \quad \vec{V} = -R\sin t\,\hat{i} + R\cos t\,\hat{j}$$

At $t=0$, $\vec{V} = r\hat{j}$, at $t=\frac{\pi}{2}$, $\vec{V} = -r\hat{i}$



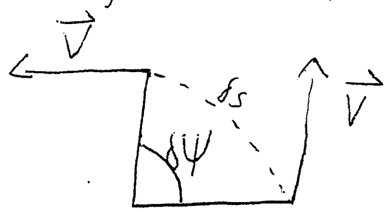
The distance traveled is represented by the dashed line. The length of this curve, $s(x,y,t)$, is the integral of the length of the velocity vector in that time increment

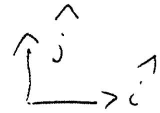
$$(3) \quad \delta s = \int |\vec{V}|\,dt$$

From (3), it follows that $|\vec{V}| = \lim_{\delta t \to 0} \frac{\delta s}{\delta t}$

(4) $$\boxed{|\vec{V}| = \frac{ds}{dt}}$$

We can also measure the rate at which $\vec{V}$ turns as it moves along the curve





by measuring the change in ψ (the direction angle that $\vec{V}$ makes with $\hat{i}$) with respect to s. At each point, the value of $\frac{d\psi}{ds}$, measured in radians per unit of length along the curve, is called the curvature. The usual notation is K.

(5) $$\boxed{\text{Curvature} \equiv K = \frac{d\psi}{ds}}$$

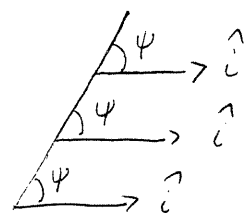
By referring to any standard calculus book, K may be computed as (Thomas and Finney, pg 772),

(6) $$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

(7) $$K = \frac{\left|\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{dy}{dt}\right)\left(\frac{d^2x}{dt^2}\right)\right|}{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}$$

Example: Show that the curvature of a straight line is zero

Solution: On a straight line, ψ has a constant value



Therefore, $\frac{d\psi}{ds} = 0$ and $K = \frac{d\psi}{ds} = 0$

Example: Show that the curvature of a circle is R^{-1}.

Solution: From 2

$$\frac{dx}{dt} = -R\sin t \qquad \frac{dy}{dt} = R\cos t$$

$$\frac{d^2x}{dt^2} = -R\cos t \qquad \frac{d^2y}{dt^2} = -R\sin t$$

$$K = \frac{|(-R\sin t)(-R\sin t) - (R\cos t)(-R\cos t)|}{[R^2\sin^2 t + R^2\cos^2 t]^{\frac{3}{2}}} = \frac{R^2}{R^3}$$

8 $\boxed{K = \frac{1}{R}}$

For a straight line, $R \to \infty$

Unit tangent and normal vectors

The unit tangent vector to the position vector $\vec{r}(x, y, t)$ is

$$(9) \quad \boxed{\hat{\tilde{\iota}} = \frac{d\vec{r}}{ds}}$$

where $\hat{\tilde{\iota}}$ is oriented parallel to the horizontal velocity trajectory at each point.

Given this definition, $\vec{V}$ may be defined as

$$(10) \quad \boxed{\vec{V} = |\vec{V}|\,\hat{\tilde{\iota}}}$$

From (2) $\hat{\tilde{\iota}}$ may be computed along a circle as

$$\hat{\tilde{\iota}} = \frac{\vec{V}}{|\vec{V}|}$$

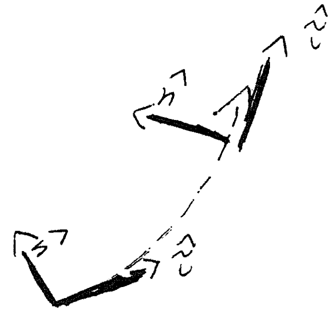
$$|\vec{V}| = \sqrt{(R\sin t)^2 + (R\cos t)^2} = \sqrt{R^2(\sin^2 t + \cos^2 t)} = R$$

Therefore

$$(11) \quad \hat{\tilde{\iota}} = \frac{\vec{V}}{|\vec{V}|} = \frac{-R\sin t\,\hat{i} + R\cos t\,\hat{j}}{R} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

If one is going to define a coordinate parallel to $\vec{V}$, it makes sense to also define a coordinate perpendicular to $\vec{V}$ at each point.

Therefore, let's define $\hat{n}$ to be oriented normal to $\vec{V}$ at each point. Let's further define $\hat{n}$ as positive left of flow direction.



Can we relate $\hat{n}$ to $\hat{\tau}$?

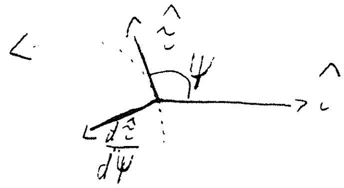
$\hat{\tau}$ may be written in terms of ψ (consistent with 11)

$$\hat{\tau} = \hat{i}\cos\psi + \hat{j}\sin\psi$$

Therefore

$$\frac{d\hat{\tau}}{d\psi} = -\hat{i}\sin\psi + \hat{j}\cos\psi = \hat{i}\cos\left(\psi + \frac{\pi}{2}\right) + \hat{j}\sin\left(\psi + \frac{\pi}{2}\right)$$

from which it follows that $\frac{d\hat{\tau}}{d\psi}$ is the unit vector obtained by rotating $\hat{\tau}$ counterclockwise through $\frac{\pi}{2}$ radians. Thus, $\frac{d\hat{\tau}}{d\psi}$ is normal to the curve at all times.



From the chain rule, $\frac{d\hat{i}}{ds} = \frac{d\hat{i}}{d\psi}\frac{d\psi}{ds}$, therefore its magnitude is

$$\left|\frac{d\hat{i}}{ds}\right| = \left|\frac{d\hat{i}}{d\psi}\right|\left|\frac{d\psi}{ds}\right| = (1)(K) = K$$

The vector $\frac{d\hat{i}}{ds}$ is normal to the curve because it is a scalar multiple of $\frac{d\hat{i}}{d\psi}$

The unit vector $\hat{n}$ is obtained by

(12) $$\boxed{\hat{n} = \frac{\left(\frac{d\hat{i}}{ds}\right)}{\left|\frac{d\hat{i}}{ds}\right|} = \frac{1}{K}\frac{d\hat{i}}{ds}}$$ (Thomas and Finney, pg 775)

$\hat{n}$ may be rewritten as

$$\hat{n} = R\,\frac{d\hat{i}/dt}{ds/dt} = R\,\frac{d\hat{i}/dt}{|\vec{V}|}$$

or

(13) $$\boxed{\frac{d\hat{i}}{dt} = \frac{|\vec{V}|}{R}\,\hat{n}}$$

Equations of motion in natural coordinates

Since $\vec{V} = |\vec{V}|\hat{\tau}$ (Eq 10)

(14) $$\frac{D\vec{V}}{Dt} = \hat{\tau}\frac{D|\vec{V}|}{Dt} + |\vec{V}|\underbrace{\frac{D\hat{\tau}}{Dt}}_{\frac{|\vec{V}|}{R}\hat{n}}$$ from (13)

or

(15) $$\frac{D\vec{V}}{Dt} = \hat{\tau}\frac{D|\vec{V}|}{Dt} + \hat{n}\frac{|\vec{V}|^2}{R}$$

And

$$\frac{D\vec{V}}{Dt} = -f\hat{k}\times\vec{V} - \nabla\Phi$$

The Coriolis force always acts normal to the direction of motion, thus

(16) $$-f\hat{k}\times\vec{V} = -f|\vec{V}|\hat{n}$$

While the pressure gradient force is

(17) $$-\nabla\Phi = -\left(\hat{\tau}\frac{\partial\Phi}{\partial s} + \hat{n}\frac{\partial\Phi}{\partial n}\right)$$

Hence

$$(18)\quad \frac{D\vec{V}}{Dt} = -f|\vec{V}|\hat{n} - \hat{t}\frac{\partial \Phi}{\partial s} - \hat{n}\frac{\partial \Phi}{\partial n} = \hat{t}\frac{D|\vec{V}|}{Dt} + \hat{n}\frac{|\vec{V}|^2}{R}$$

Therefore, the horizontal momentum equations may be written into the following component equations in natural coordinates

$$\boxed{\begin{aligned} &(19)\quad \frac{D|\vec{V}|}{Dt} = -\frac{\partial \Phi}{\partial s} \\ &(20)\quad \frac{|\vec{V}|^2}{R} + f|\vec{V}| = -\frac{\partial \Phi}{\partial n} \end{aligned}}$$

Note: $R > 0$ if air parcels turn toward the left following motion (cyclonic)

$R < 0$ if air parcels turn toward the right following motion (anticyclonic).