QG- omega equation

The equations are

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \qquad (1)$$

$$\frac{\partial T}{\partial t} = -\vec{V}_g \cdot \nabla T + \omega \sigma \frac{p}{R} \qquad (2)$$

$$T = -\frac{p}{R}\frac{\partial \Phi}{\partial p} \qquad (3)$$

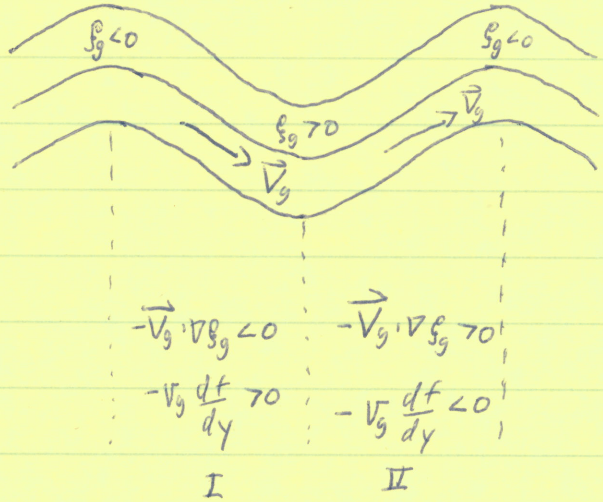
Where, in addition to the Q-G approximation, the following "extra" approximations have been made: a) no diabatic heating, or $\frac{J}{c_p} = 0$ b) no friction, and c) $f = f_0 + \beta y$, the Beta plane approximation.

Note that when no divergence occurs in (1) such that $\frac{\partial \omega}{\partial p} = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$, ζ_g may change by two processes locally: a) advection of relative vorticity, and b) advection of earth vorticity. One may be written as:

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla \zeta_g - \beta v_g + \underbrace{\cancel{f_0 \frac{\partial \omega}{\partial p}}}_{\text{assume zero for non-divergent flow}}$$

which is called the barotropic vorticity equation since $\omega = 0$ and (it turns out), $\nabla T = 0$ also

Note the following figure from Holton (Fig. 6.7, pg 154)



In region I downstream of a ridge (upstream from a trough), the geostrophic wind is directed from a negative vorticity maximum at the ridge toward the positive vorticity maximum at the trough. Therefore, $-\vec{V}_g \cdot \nabla \zeta_g < 0$. But at the same time, $v_g < 0$ in region one, and positive advection of earth vorticity is occurring $(-\beta v_g > 0)$. Hence, in region I, advection of ζ_g contributes to $\frac{\partial \zeta_g}{\partial t} < 0$, but advection of "f" contributes to $\frac{\partial \zeta_g}{\partial t} > 0$. Similar arguments (but with reversed signs) apply to region II.

Hence, relative vorticity advection tends to move troughs and ridges downstream, while earth vorticity advection moves the pattern upstream (called the "Beta effect.") The latter motion is called "retrograde motion" or "retrogression," The following statements can be made (see Holton, section 6.2.2, for more details):

Wavelength	Advection assessment	Wave motion
< 3000 km	rel. vort. adv. > Beta effect	short waves move downstream fast
> 10,000 km	rel. vort. adv. < Beta effect	long waves retrograde
between	depends	stationary or move slow to east

However, vorticity advection alone does not determine the evolution of weather systems. Where $\omega \neq 0$, ζ_g and T change in a complicated fashion. A change in $\vec{V}_g$ with height (which indicates temperature advection), and furthermore, changes in $-\vec{V} \cdot \nabla \zeta_g$ with height will result in an ageostrophic vertical circulation, which then adjusts ∇T to maintain thermal wind balance. In turn, the resulting divergence and convergence fields will force changes in the ζ_g distribution.

Furthermore, temperature advection not only changes T, but also induces ω, which in turn alters ζ_g. We can ~~[illegible]~~ combine these factors in one equation, called the "omega equation," which is used qualitatively in map analysis.

The motivation behind QG theory is that ~~they~~ it is ~~[illegible]~~ approximately valid for synoptic-scale systems, and is easier to interpret than the more complicated primitive equations. In weather analysis, "synoptic forcing" ~~[illegible]~~ generally drives smaller-scale weather features. Even if frontal ascent is occurring, or if the atmosphere is unstable, descent driven by the synoptic scale will overcome these local features, and no "rainfall will occur.

There is a "philosophy" behind Q-G theory as well which aids our intrepetation of weather analysis. The Q-G philosophy is: "Geostrophic advection brings the atmosphere out of hydrostatic and geostrophic balance, and ageostrophic processes bring the atmosphere back into balance." through vertical motion.

Derivation of omega equation

Since $u_g = \frac{1}{f_0}\frac{\partial \Phi}{\partial y}$, $v_g = -\frac{1}{f_0}\frac{\partial \Phi}{\partial x}$, ζ_g is:

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0}\nabla^2 \Phi \qquad (4)$$

By substituting (4) into (1), and (3) into (2), one obtains:

$$\frac{1}{f_0}\nabla^2 \chi = -\vec{V}_g \cdot \nabla(\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \qquad (5)$$

$$-\frac{p}{R}\frac{\partial \chi}{\partial p} = -\vec{V}_g \cdot \nabla T + \omega \sigma \frac{p}{R} \qquad (6)$$

where $\chi = \frac{\partial \Phi}{\partial t}$, and is called the "geopotential height tendency." ~~"geopotential height tendency"~~

We want to eliminate χ and obtain an expression for ω. The following step is performed:

$$\frac{R}{p\sigma}\nabla^2 (6) - \frac{f_0}{\sigma}\frac{\partial}{\partial p} (5)$$

and using the identity:

$$\nabla^2(\sigma\omega) = \omega\nabla^2\sigma + 2\,\nabla\sigma\cdot\nabla\omega + \sigma\nabla^2\omega$$

the following equation occurs:

$$\left[\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2} + \left(\frac{\nabla^2\sigma}{\sigma} + \frac{2}{\sigma}\nabla\sigma\cdot\nabla\right)\right]\omega =$$

$$-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla(\zeta_g + f)\right] - \frac{R}{\sigma p}\nabla^2\left(-\vec{V}_g\cdot\nabla T\right)$$

Holton now makes another "extra" approximation that $\nabla\sigma = 0$ (σ is horizontally uniform), ~~and thus~~: which also means $\nabla^2\sigma = 0$. The remaining equation is:

$\cancel{\nabla^2\sigma = 0}$

$$\boxed{\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla(\zeta_g + f)\right] - \frac{R}{\sigma p}\nabla^2\left(-\vec{V}_g\cdot\nabla T\right)}$$

called the "QG omega equation."