Orthogonality relationships used in Fourier series

Consider the cases where $m \neq n$

$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \int_0^{2\pi} \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right]dx$$

$$= \left(\frac{\sin[(m-n)x]}{2(m-n)} - \frac{\sin[(m+n)x]}{2(m+n)}\right)\Bigg]_0^{2\pi}$$

$$= \emptyset \;!!!$$ since all sin multiples of 2π and zero are zero

Some books instead integrate from $-\pi$ to π. This will also be zero. In this case, all sin multiples of π are zero. Physically, this means the positive areas and negative areas of a periodic wave cancels.

Likewise

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx = \int_0^{2\pi} \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right]dx$$

$$= \left(\frac{\sin[(m-n)x]}{2(m-n)} + \frac{\sin[(m+n)x]}{2(m+n)}\right)\Bigg]_0^{2\pi}$$

$$= \emptyset \;!!!$$

$$\int_0^{2\pi} \sin(mx)\cos(nx)\,dx = \int_0^{2\pi} \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right]dx$$

$$= \left(-\frac{\cos[(m-n)x]}{2(m-n)} - \frac{\cos[(m+n)x]}{2(m+n)}\right)\Bigg]_0^{2\pi}$$

$$= \emptyset\ !!!$$

Relationships where two functions multiplied together, which are then zero after integration, are known as orthogonal relationships.

For situations when $m=n$, let's take the above situation first:

$$\int_0^{2\pi} \sin(nx)\cos(nx)\,dx = -\frac{\cos[(2n)x]}{4n}\Bigg]_0^{2\pi} = \emptyset\ !!!$$

So, even for $m=n$, $\int_0^{2\pi} \sin(mx)\cos(nx) = 0$.
This relationship is orthogonal for all n and m,

Let's check $m=n$ for multiples of sin and cos

$$\int_0^{2\pi} \sin(nx)\sin(nx)\,dx = \int_0^{2\pi} \frac{1}{2}\Big[\underbrace{\cos(n-n)}_{\substack{\cos(0)\\=1}}x - \underbrace{\cos(n+n)}_{\cos(2n)}x\Big]dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2n)x}{4n}\right]\Bigg]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi - 0 - 0 + 0\right] = \pi$$

$$\int_0^{2\pi} \cos(nx)\cos(nx)\,dx = \int_0^{2\pi} \frac{1}{2}\Big[\underbrace{\cos(n-n)}_{\cos(0)}x + \underbrace{\cos(n+n)}_{\cos(2n)}x\Big]dx$$

$$= \pi$$

The fact that only certain multiples of trig functions can be integrated to a non-zero value allows us to extract the important wave frequencies from a time series or $\Psi(x)$.

Let's now return to the $\frac{2\pi nx}{L}$ space. The following orthogonality relationships apply:

$$\int_0^L \sin\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}\,dx = 0 \quad \text{for all } m \text{ and } n$$

$$\int_0^L \sin\frac{2\pi nx}{L}\sin\frac{2\pi mx}{L}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ L/2 & \text{if } n = m \end{cases}$$

$$\int_0^L \cos\frac{2\pi nx}{L}\cos\frac{2\pi mx}{L}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ L/2 & \text{if } n = m \end{cases}$$