The Vorticity Equation

We can derive a prognostic equation for ζ_a by starting with the horizontal equations of motion in (x, y, z) coordinates:

$$\text{1} \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\text{2} \qquad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Since $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, then $\frac{\partial}{\partial x}$ 2 $- \frac{\partial}{\partial y}$ 1 will yield the "vorticity equation."

However, this involves many steps. First

$$\frac{\partial}{\partial x}\,\text{2} \Rightarrow \frac{\partial}{\partial t}\frac{\partial v}{\partial x} + \vec{V}\cdot\nabla\frac{\partial v}{\partial x} + \frac{\partial \vec{V}}{\partial x}\cdot\nabla v + f\frac{\partial u}{\partial x} + \overset{\to 0}{\cancel{u\frac{\partial f}{\partial x}}} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} + \frac{1}{\rho^2}\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}$$

$$\frac{\partial}{\partial y}\,\text{1} \Rightarrow \frac{\partial}{\partial t}\frac{\partial u}{\partial y} + \vec{V}\cdot\nabla\frac{\partial u}{\partial y} + \frac{\partial \vec{V}}{\partial y}\cdot\nabla u - f\frac{\partial v}{\partial y} - v\frac{\partial f}{\partial y} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} + \frac{1}{\rho^2}\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}$$

And $\frac{\partial}{\partial x}$ 2 $- \frac{\partial}{\partial y}$ 1 gives

$$\text{3} \qquad \frac{\partial \zeta}{\partial t} + \vec{V}\cdot\nabla\zeta + \frac{\partial \vec{V}}{\partial x}\cdot\nabla v - \frac{\partial \vec{V}}{\partial y}\cdot\nabla u + f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{\partial f}{\partial y} = \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

It can be shown that:

$$\frac{\partial \vec{V}}{\partial x} \cdot \nabla v - \frac{\partial \vec{V}}{\partial y} \cdot \nabla u = \frac{\partial u}{\partial x}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\zeta} + \frac{\partial v}{\partial y}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\zeta}$$

$$+ \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$$

or, combining terms

(4) $$\frac{\partial \vec{V}}{\partial x} \cdot \nabla v - \frac{\partial \vec{V}}{\partial y} \cdot \nabla u = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\zeta + \frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}$$

Also note that

(5) $$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \vec{V} \cdot \nabla \zeta$$

(6) $$\frac{Df}{Dt} = \cancelto{0}{\frac{\partial f}{\partial t}} + u\cancelto{0}{\frac{\partial f}{\partial x}} + v\frac{\partial f}{\partial y} + w\cancelto{0}{\frac{\partial f}{\partial z}}$$

Plug (4), (5), (6) into (3) gives, upon rewriting

(7) $$\boxed{\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\text{convergence term}} - \underbrace{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{\text{tilting term}} + \frac{1}{\rho^2}\underbrace{\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{\text{solenoid term}}}$$

Eq 7 may be rewritten using the following relationships:

$$\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} = \frac{\partial w}{\partial y}\left(\eta + \frac{\partial w}{\partial x}\right)$$

$$\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} = \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial y} - \xi\right)$$

Since $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$. Thus

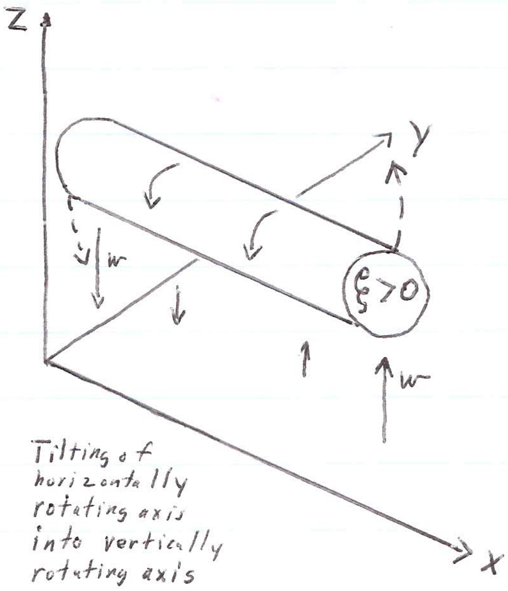
$$\boxed{\frac{D(\zeta+f)}{Dt} = -(\zeta+f)\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{1} + \underbrace{\left(\xi\frac{\partial w}{\partial x} + \eta\frac{\partial w}{\partial y}\right)}_{2} + \underbrace{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{3}} \quad (8)$$

Interpretation:

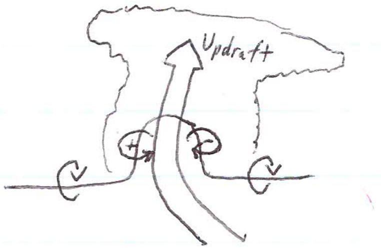
1) Horizontal convergence ($\nabla_H \cdot \vec{V} < 0$) cause ζ_a to become more cyclonic, or divergence cause ζ_a to become less cyclonic. Note that vertical stretching of an air column increases its cyclonic vorticity (conservation of angular momentum principal).

2) Vertical vorticity is generated by tilting horizontal vorticity components into the vertical by a nonuniform vertical motion field (w varies horizontally)

Conversely, vertical shear of the horizontal velocity creates η or ξ by tilting vertical vortices.



Tilting of horizontally rotating axis into vertically rotating axis

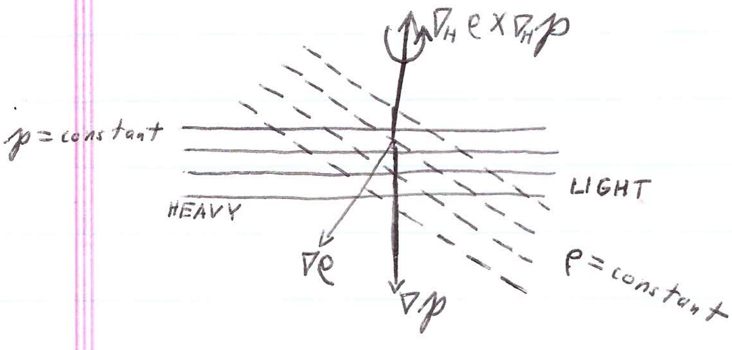


Development of rotating updrafts (mesocyclones)

(3) This describes how baroclinicity is going to contribute to vorticity of the wind. First, the following can be shown:

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) = \hat{k} \cdot \frac{\nabla_H \rho \times \nabla_H p}{\rho^2}$$

$$= \hat{k} \cdot \left(\nabla_H p \times \nabla_H T\right)\frac{R}{p}$$



Example:

In this case, $\nabla\rho$ and ∇p are constant vectors. The lighter fluid on the right feels the same $-\nabla p$ as the heavier fluid on the left.

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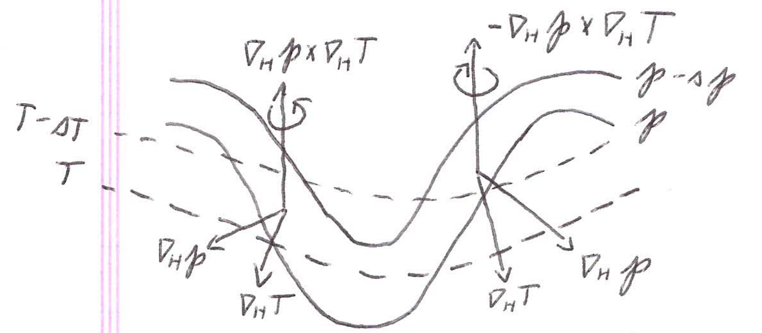
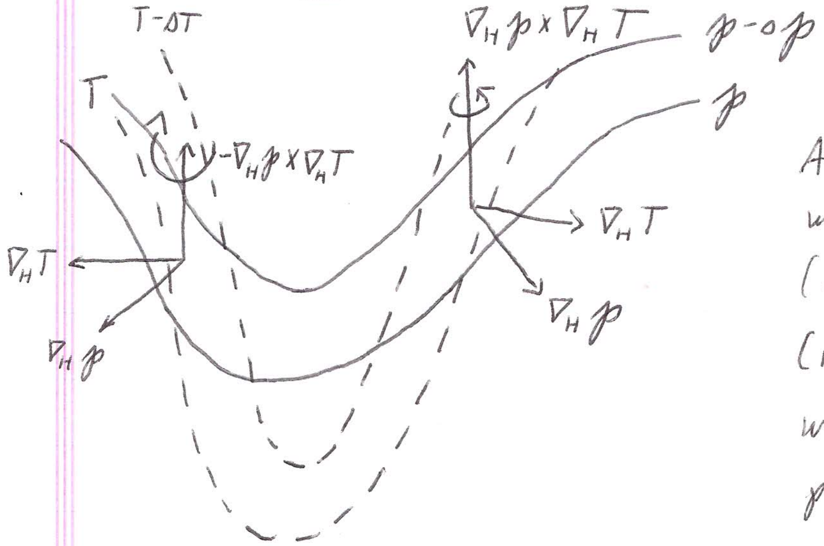
The lighter fluid will therefore (in the absence of other effects) tend to move northward more rapidly than the heavier fluid, resulting in a net cyclonic circulation.

Note that the resulting circulation will tend to align the density and pressure surfaces. In other words, the increase of ζ is associated with a reduction of baroclinicity. In other words, the atmosphere is always trying to bring itself back into equilibrium, with a tendency toward a barotropic atmosphere. PE => KE

Another example:



Air moving through the wave will acquire cyclonic vorticity (anticyc vort) as it approaches trough (ridge). Trough and ridge will strengthen. ∇T will weaken.



Air moving through the wave will acquire anticyclonic vorticity (cyc vort) as it approaches trough (ridge). Trough and ridge will weaken. ∇T will increase, possibly forming a "cold pocket."

Note that in a barotropic atmosphere, where surfaces of p and ρ coincide,

$$\hat{k} \cdot \nabla_H \rho \times \nabla_H p = 0$$

and

$$\hat{k} \cdot \nabla_H p \times \nabla_H T = 0$$

The solenoid term is zero.

Vorticity equation in isobaric coordinates

Since $PGF = -\nabla \Phi$, there is no density term, and the equation in (x, y, p) is simpler

(9) $$\boxed{\frac{D}{Dt}(\zeta + f)_p = -(\zeta + f)_p \nabla_p \cdot \vec{V} + \left(\frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p} - \frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} \right)_p}$$

where the subscript p emphasizes that differentiation is to be performed on an isobaric surface. Note: no solenoid term!

Scale analysis

On the synoptic scale, the approximate vorticity equation is

(10) $$\boxed{\frac{D_H(\zeta + f)}{Dt} = -f \, \nabla_H \cdot \vec{V}} \, ; \quad \frac{D_H}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Not valid in fronts, thunderstorms. Somewhat incorrect in highs and lows.