Lower-Order Spectral Model

5.1 Introduction

This system was first developed by Lorenz (1960b). It is an elegant system that provides an introduction to the concepts of spectral modeling, based on the use of double Fourier series representations of the basic equations in a doubly periodic domain. Here we examine the *barotropic vorticity equation.* We start with the equation governing the conservation of vorticity of a parcel for two-dimensional, homogeneous, incompressible, and inviscid fluid flow on an f-plane given by

$$\frac{\partial}{\partial t}\nabla^2\psi = -J\left(\psi, \nabla^2\psi\right), \quad \text{or} \quad \frac{\partial}{\partial t}\nabla^2\psi = -\vec{k}\cdot\nabla\psi\times\nabla\left(\nabla^2\psi\right). \tag{5.1}$$

Since we are working on an f-plane, the β term does not appear in this equation. ψ is the streamfunction and J is the Jacobian.

We state the periodicity property by the relation

$$\psi(x,y,t) = \psi\left(x + \frac{2\pi}{k}, y + \frac{2\pi}{l}, t\right),$$

where k and l are constants. We seek a solution to (5.1) in a closed horizontal domain. For this, we expand ψ following Lorenz (1960b), that is,

$$\psi = \sum_{m=0}^{\infty}\sum_{n=n_0}^{\infty}\frac{-1}{m^2k^2+n^2l^2}\left[A_{mn}\cos(mkx+nly) + B_{mn}\sin(mkx+nly)\right]. \tag{5.2}$$

This is the *double Fourier representation* of the function ψ. The coefficients A_{mn} and B_{mn} are functions of time, where m and n are integers representing east–west and north–south wavenumbers, respectively. Note that $A_{00} = 0$. In addition, the lower limit n_0 of n is specified as

$$n_0 = \begin{cases} -\infty & \text{if} \quad m > 0 \\ 0 & \text{if} \quad m = 0 \end{cases} .$$

The series will now be truncated and we consider only those terms for which m equals 0 and +1 and n equals -1, 0, and +1; in other words, we include only one wave in both directions. Equation (5.2) for the streamfunction ψ then reduces to

$$\begin{aligned} \psi = &-\frac{A_{10}}{k^2}\cos kx - \frac{A_{01}}{l^2}\cos ly - \frac{A_{11}}{k^2+l^2}\cos(kx+ly) \\ &- \frac{A_{1,-1}}{k^2+l^2}\cos(kx-ly) - \frac{B_{10}}{k^2}\sin kx - \frac{B_{01}}{l^2}\sin ly \\ &- \frac{B_{11}}{k^2+l^2}\sin(kx+ly) - \frac{B_{1,-1}}{k^2+l^2}\sin(kx-ly) . \end{aligned} \tag{5.3}$$

The corresponding relative vorticity is given by

$$\begin{aligned} \nabla^2\psi = &A_{10}\cos kx + A_{01}\cos ly + A_{11}\cos(kx+ly) \\ &+ A_{1,-1}\cos(kx-ly) + B_{10}\sin kx \\ &+ B_{01}\sin ly + B_{11}\sin(kx+ly) + B_{1,-1}\sin(kx-ly) . \end{aligned} \tag{5.4}$$

Substituting the Fourier expansion of ψ and $\nabla^2\psi$ into (5.1) and taking the Fourier transform of both sides of the resulting equation, we get the prediction equations for the amplitude of the different wave components. In all, we have eight equations providing time tendencies for each of the eight amplitudes.

5.2 Maximum Simplification

After substituting for ψ and $\nabla^2\psi$ from (5.3) and (5.4) into (5.1) and equating the coefficients of the various Fourier functions on both sides of the resulting equation, we get a set of differential equations for the coefficients A_{10}, A_{01}, A_{11}, A_{1-1}, B_{10}, B_{01}, B_{11}, and B_{1-1}. Following Lorenz (1960b), we assume: (a) If B_{10}, B_{01}, B_{11}, and B_{1-1} vanish initially, then they will remain zero for all time since their tendencies are always equal to zero, that is,

$$\frac{dB_{10}}{dt} = \frac{dB_{01}}{dt} = \frac{dB_{11}}{dt} = \frac{dB_{1-1}}{dt} = 0 .$$

We thus obtain $B_{10} = B_{01} = B_{11} = B_{1-1} = 0$. (b) If $A_{1-1} = -A_{11}$ initially, then A_{1-1} will remain equal to $-A_{11}$ for all time. Furthermore, let $A_{01} = A$, $A_{10} = F$, and $-A_{11} = G$. With this, (5.3) reduces to

$$\psi = -\frac{A}{l^2}\cos ly - \frac{F}{k^2}\cos kx - 2\frac{G}{k^2+l^2}\sin kx \sin ly . \tag{5.5}$$

A, F, and G are functions of time. The term $-(A/l^2)\cos ly$ describes the basic zonal current, that is, it has no x dependence. The term $-(F/k^2)\cos kx - [2G/(k^2+l^2)]\sin kx \sin ly$ describes the eddies. The corresponding relative vorticity is given by

$$\nabla^2\psi = A\cos ly + F\cos kx + 2G\sin kx \sin ly\,. \tag{5.6}$$

Next we obtain an expression for

$$J\left(\psi, \nabla^2\psi\right) = \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x}\,. \tag{5.7}$$

Now,

$$\frac{\partial\psi}{\partial x} = \frac{F}{k}\sin kx - \frac{2Gk}{k^2+l^2}\cos kx \sin ly\,, \tag{5.8}$$

$$\frac{\partial\psi}{\partial y} = \frac{A}{l}\sin ly - \frac{2Gl}{k^2+l^2}\sin kx \cos ly\,, \tag{5.9}$$

$$\frac{\partial}{\partial x}\nabla^2\psi = -Fk\sin kx + 2Gk\cos kx \sin ly\,, \tag{5.10}$$

$$\frac{\partial}{\partial y}\nabla^2\psi = -Al\sin ly + 2Gl\sin kx \cos ly\,. \tag{5.11}$$

Hence

$$\begin{aligned} J\left(\psi, \nabla^2\psi\right) = &\left(\frac{F}{k}\sin kx - \frac{2Gk}{k^2+l^2}\sin ly \cos kx\right) \\ &\times\left(-Al\sin ly + 2Gl\sin kx\cos ly\right) \\ &-\left(\frac{A}{l}\sin ly - \frac{2Gl}{k^2+l^2}\cos ly \sin kx\right) \\ &\times\left(-Fk\sin kx + 2Gk\cos kx\sin ly\right). \end{aligned}$$

After simplifying, we obtain

$$\begin{aligned} J\left(\psi, \nabla^2\psi\right) = &\left(\frac{1}{l^2} - \frac{1}{k^2}\right) AFkl\sin kx\sin ly \\ &+\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right) 2FGkl\sin^2 kx\cos ly \\ &+\left(-\frac{1}{l^2} + \frac{1}{k^2+l^2}\right) 2AGkl\sin^2 ly\cos kx\,. \end{aligned} \tag{5.12}$$

Differentiating (5.6) with respect to t, we get

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{dA}{dt}\cos ly + \frac{dF}{dt}\cos kx + 2\frac{dG}{dt}\sin kx\sin ly\,. \tag{5.13}$$

From the barotropic vorticity equation (5.1) along with (5.12) and (5.13), we get

$$
\begin{aligned}
&\frac{dA}{dt}\cos ly + \frac{dF}{dt}\cos kx + 2\frac{dG}{dt}\sin kx \sin ly \\
&\quad = -\left[\left(\frac{1}{l^2} - \frac{1}{k^2}\right) AFkl \sin kx \sin ly \right. \\
&\qquad + \left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right) 2FGkl \sin^2 kx \cos ly \\
&\qquad \left. + \left(-\frac{1}{l^2} + \frac{1}{k^2+l^2}\right) 2AGkl \sin^2 ly \cos kx \right]. \qquad (5.14)
\end{aligned}
$$

If we multiply (5.14) by $\cos ly$ and integrate both sides over the entire doubly periodic fundamental domain, then using the orthogonality properties of the Fourier functions we obtain

$$
\begin{aligned}
\frac{dA}{dt}\int_0^{2\pi}\int_0^{2\pi} \cos^2 ly \, dx\, dy &= -\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right) 2klFG \\
&\quad \times \int_0^{2\pi}\int_0^{2\pi} \sin^2 kx \cos^2 ly \, dx\, dy \,. \qquad (5.15)
\end{aligned}
$$

Integrating, we obtain

$$2\frac{dA}{dt}\pi^2 = -2\pi^2 klFG\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right),$$

or

$$\frac{dA}{dt} = -\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right) klFG\,. \qquad (5.16)$$

Similarly, if we multiply (5.14) by $\cos kx$ and $\sin kx \sin ly$ and integrate over the domain, we get

$$\frac{dF}{dt} = \left(\frac{1}{l^2} - \frac{1}{k^2+l^2}\right) klAG\,, \qquad (5.17)$$

$$\frac{dG}{dt} = -\frac{1}{2}\left(\frac{1}{l^2} - \frac{1}{k^2}\right) klAF\,. \qquad (5.18)$$

Equations (5.16), (5.17), and (5.18) are a system of three coupled nonlinear first-order ordinary differential equations in the three unknowns A, F, and G. If their initial values are known, then their future values can be obtained using numerical integration. The above system has exact solutions which can be expressed by elliptic functions (or circular functions) in time.