Runga Kutta Methods

Many equations of the form:

$$\frac{Dx}{Dt} = f(t,x)$$

solve for x on computers using Runga Kutta (RK) methods. There are a variety of RK schemes which vary in accuracy. The simplest (and often accurate enough for "back of the envelope" calculations) is the RK order 2 method which involves guessing an initial solution by:

$$x_{guess} = x^{\tau} + \Delta t\, f(t_o, x^{\tau})$$

where Δt is the time step, t_o is the time value at the beginning of the time step, and x^{τ} is the initial x value. Then, this solution is plugged into the equation again to obtain the future value of x at $t_o + \Delta t$ (which we'll call $x^{\tau+1}$):

$$x^{\tau+1} = x^{\tau} + \frac{\Delta t}{2}\left[f(t_o, x^{\tau}) + f(t_o + \Delta t, x_{guess})\right]$$

If one then wants to predict an x value even further into the future, set $x^{\tau+1} = x^{\tau}$ and do another calculation. (The RK order 2 method is also sometimes called the *Improved Euler Method*, the *midpoint method*, or the *predictor-corrector method*).

There is no reason to stop there. To improve accuracy even more, one may substitute the second calculation into the equation again, compute x, insert it into the equation, compute x, etc. If one does this four times, one obtain the following equation (after algebraically combining common terms):

$$k_1 = \Delta t\, f(t_o, x^{\tau})$$
$$k_2 = \Delta t\, f(t_o + \frac{\Delta t}{2}, x^{\tau} + \frac{k_1}{2})$$
$$k_3 = \Delta t\, f(t_o + \frac{\Delta t}{2}, x^{\tau} + \frac{k_2}{2})$$
$$k_4 = \Delta t\, f(t_o + \Delta t, x^{\tau} + k_3)$$
$$x^{\tau+1} = x^{\tau} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

This is called the RK method of order 4, and is the most popular.