12.5 Semi-Lagrangian methods

Ideally, one should be able to integrate the advection equation by following the fluid particles in a Lagrangian manner, so that the local rate of change and advection terms do not have to be considered separately. In fact, taking a *Lagrangian approach*, a graphical method has been developed to solve the barotropic vorticity equation using a single time step of 24 h by following a set of fluid particles (Fjortoft 1952). However, in general a set of fluid particles, which are initially distributed regularly, will soon become greatly deformed and are thus rendered unsuitable for numerical integration (Welander 1955). To avoid this difficulty, the *semi-Lagrangian method* (occasionally referred to as *quasi-Lagrangian method*) whereby a set of particles that arrive at a regular set of grid points are traced backward over a single time step to their departure points was proposed (Wiin-Nielsen 1959). The values of the dynamical quantities at the departure points are obtained by interpolating known values at neighboring grid points. Note that in a semi-Lagrangian method, the set of fluid particles in question changes at each time step, which is different from the pure Lagrangian method. In addition, a combination of these schemes, i.e. *semi-Lagrangian semi-implicit scheme*, has been proposed (Robert 1982; Staniforth and Côté 1991).

To examine the stability property of the semi-Lagrangian method, we consider the one-dimensional nonlinear advection equation in the form of total derivative,

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = 0, \qquad (12.5.1)$$

where $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + u\partial/\partial x$ and ψ is any variable under consideration. By integrating over the trajectory of a fluid particle that arrives at a grid point $i\Delta x$, denoted as P in Fig. 12.9, and at time $(\tau+1)\Delta t$, we have

$$\psi_i^{\tau+1} = \psi_*^{\tau}, \qquad (12.5.2)$$

where ψ_*^{τ} is the value of ψ at the departure point of the particle at time $\tau\Delta t$. The value ψ_*^{τ} is obtained by polynomial interpolation from the neighboring grid points. The stability and accuracy of the scheme depends on the interpolation method used. For example, we may consider the linear interpolation from the surrounding grid points $(i-p)$ and $(i-p-1)$ for ψ_*^{τ},

$$\frac{\psi_{i-p}^{\tau} - \psi_*^{\tau}}{u\Delta t - p\Delta x} = \frac{\psi_{i-p}^{\tau} - \psi_{i-p-1}^{\tau}}{\Delta x}, \qquad (12.5.3)$$

where u is the advection velocity as represented in (12.5.1). The above equation may be rearranged as

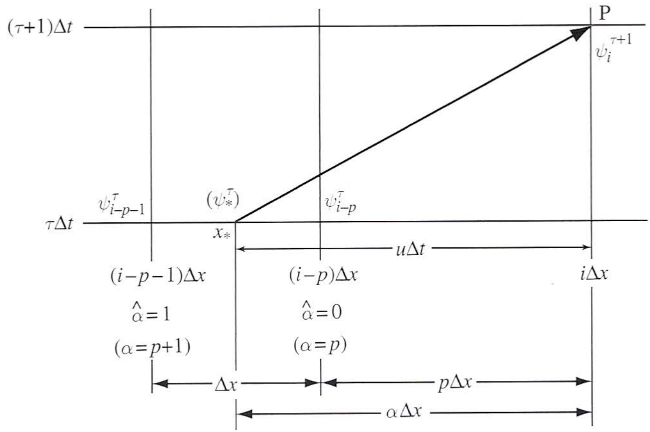

Fig. 12.9 A schematic of the semi-Lagrangian method. A fluid particle that arrives at a grid point $i\Delta x$ and at time $(\tau+1)\Delta t$ is denoted as P, which is located at x_* and at time $\tau\Delta t$. The value of the variable at this time and location (ψ_*^{τ}) is obtained by polynomial interpolation from the neighboring grid points, ψ_{i-p-1}^{τ} and ψ_{i-p}^{τ}, as expressed in (12.5.4) or (12.5.5).

$$\psi_*^{\tau} = \psi_{i-p}^{\tau} - \left(\frac{u\Delta t}{\Delta x} - p\right)\left(\psi_{i-p}^{\tau} - \psi_{i-p-1}^{\tau}\right). \tag{12.5.4}$$

or

$$\psi_*^{\tau} = \psi_{i-p}^{\tau} - \hat{\alpha}(\psi_{i-p}^{\tau} - \psi_{i-p-1}^{\tau}), \tag{12.5.5}$$

where

$$\hat{\alpha} = \alpha - p, \quad \alpha = u\Delta t/\Delta x. \tag{12.5.6}$$

Therefore, from (12.5.2) we have

$$\psi_i^{\tau+1} = \psi_{i-p}^{\tau} - \hat{\alpha}(\psi_{i-p}^{\tau} - \psi_{i-p-1}^{\tau}). \tag{12.5.7}$$

According to (12.5.6) and Fig. 12.9, $\hat{\alpha}$ is the fractional part, and p is the integral part after advection of a non-dimensional distance $u\Delta t/\Delta x$.