2.5 Time differencing schemes

We consider time differencing schemes first in the context of ODE's of the type

$$\frac{dU}{dt} = f(U,t), \quad U = U(t), \tag{2.29}$$

where t denotes the time. We divide the time axis into segments of equal length Δt and denote the approximate value of U at time $\tau\,\Delta t$ by $U^{(\tau)}$. We assume that $U^{(0)}$, $U^{(1)}$, ... , $U^{(n)}$ are known and seek to construct a scheme for computation of an approximate value $U^{(\tau+1)}$. There are many possibilities.

2.5.1 Two-level schemes

These relate $U^{(\tau+1)}$ to $U^{(\tau)}$ by approximating the exact formula

$$U^{(\tau+1)} = U^{(\tau)} + \int_{\tau\,\Delta t}^{(\tau+1)\Delta t} f(U,t)dt. \tag{2.30}$$

Euler (or forward) scheme: In this, $f(U,t)$ is assumed constant, equal to the value at time $\tau\,\Delta t$. Then

$$U^{(\tau+1)} = U^{(\tau)} + \Delta t\, f^{(\tau)},$$

where

$$f^{(\tau)} = f(U^{(\tau)}, \tau\,\Delta t). \tag{2.31}$$

The truncation error of this scheme is $O(\Delta t)$, i.e., it is a first-order accurate scheme. Because f is computed at one end of the interval, the scheme is said to be *uncentred* in time. Such schemes are invariably only first-order accurate.

Backward scheme: Now $f(U,t)$ is assumed constant, equal to $f^{(\tau+1)}$, i.e.

$$U^{(\tau+1)} = U^{(\tau)} + \Delta t\, f^{(\tau+1)}. \tag{2.32}$$

Since $f^{(\tau+1)}$ depends on $U^{(\tau+1)}$, the scheme is called *implicit*. For an ODE it may be simple to solve such a difference equation for $U^{(\tau+1)}$, but for PDE's this requires solving a set of simultaneous equations, one for each grid point of the computational region.

If the value of $f^{(\tau)}$ does not depend on $U^{(\tau+1)}$, such as in (2.31), the scheme is called *explicit*.

The truncation error of (2.32) is also $O(\Delta t)$.

Trapezoidal scheme: Here f is assumed to vary linearly between its values at $\tau\,\Delta t$ and $(\tau+1)\Delta t$ so that its mean value is used for the integral, i.e.

$$U^{(\tau+1)} = U^{(\tau)} + \tfrac{1}{2}\Delta t\,(f^{(\tau+1)} + f^{(\tau)}). \tag{2.33}$$

This is an implicit scheme also, but its truncation error is $O(\Delta t^2)$.

Next we consider a pair of iterative schemes:

Matsuno (or Euler-backward) schemes: First a step is made using the Euler scheme. The value of $U_*^{\tau+1}$ so obtained is used to approximate $f^{(\tau+1)}$, say $f_*^{(\tau+1)}$, which is then used to make a backward step. Therefore

$$\left.\begin{aligned} U_*^{(\tau+1)} &= U^{(\tau)} + \Delta t\, f^{(\tau)}, \\ U^{(\tau+1)} &= U^{(\tau)} + \Delta t\, f_*^{(\tau+1)}, \end{aligned}\right\}, \tag{2.34}$$

where

$$f_*^{(\tau+1)} = f(U_*^{(\tau+1)}, (\tau+1)\Delta t).$$

This is an explicit scheme and is first-order accurate.

Heun scheme: This is similar to the previous one, it is explicit and second-order accurate, but the second step is made using the trapezoidal scheme, i.e.

$$\left.\begin{aligned} U_*^{(\tau+1)} &= U^{(\tau)} + \Delta t\, f^{(\tau)}, \\ U^{(\tau+1)} &= U^{(\tau)} + \tfrac{1}{2}\Delta t\,\left(f^{(\tau)} + f_*^{(\tau+1)}\right). \end{aligned}\right\} \tag{2.35}$$

2.5.2 Three-level schemes

Except at the first time step, one can store the value $U^{(\tau-1)}$ and construct schemes using the 'time history' of U. These are three-level schemes. They approximate the formula

$$U^{(\tau+1)} = U^{(\tau-1)} + \int_{(\tau-1)\Delta t}^{(\tau+1)\Delta t} f(U,t)\,dt, \tag{2.36}$$

or they can use the additional value to improve the approximation to f in (2.30). Two examples are:

Leapfrog scheme: In this, f is taken to be constant, equal to the value at time $\tau\,\Delta t$, whereupon

$$U^{(\tau+1)} = U^{(\tau-1)} + 2\Delta t\, f^{(\tau)}. \tag{2.37}$$

This is probably the scheme most widely used in the atmospheric sciences. It is second-order accurate with truncation error $O(\Delta t^2)$.

Adams-Bashforth scheme: The scheme that is usually called the Adams-Bashforth scheme in the atmospheric sciences is, in fact, a simplified version of the original Adams-Bashforth scheme, which is fourth-order accurate. The simplified version is obtained when f in (2.30) is approximated by a value obtained at the centre of the interval Δt by a linear extrapolation using values $f^{(\tau-1)}$ and $f^{(\tau)}$. This gives

$$U^{(\tau+1)} = U^{(\tau)} + \Delta t\left(\tfrac{3}{2} f^{(\tau)} - \tfrac{1}{2} f^{(\tau-1)}\right). \tag{2.38}$$

This is an explicit, second-order accurate scheme.

Milne-Simpson scheme: In this, Simpson's rule is used to calculate the integral in (2.36), giving an implicit scheme.