

Sound (acoustic) waves

The pertinent equations are:

$$\textcircled{I} \quad \frac{Dy}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\textcircled{II} \quad \frac{D\rho}{Dt} = -\rho \frac{\partial y}{\partial x} \quad \text{or} \quad \frac{1}{\rho} \frac{D\rho}{Dt} = -\frac{\partial y}{\partial x} \quad \text{or} \quad \frac{D \ln \rho}{Dt} = -\frac{\partial y}{\partial x}$$

$$\textcircled{III} \quad C_p \frac{DT}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} = 0$$

\textcircled{III} can be written as

$$C_v \frac{D \ln \rho}{Dt} - C_p \frac{D \ln \rho}{Dt} = 0$$

$$\text{Let } \gamma = \frac{C_p}{C_v}$$

$$\textcircled{IV} \quad \frac{1}{\gamma} \frac{D \ln \rho}{Dt} - \frac{D \ln \rho}{Dt} = 0 \quad \begin{matrix} \text{This has removed } T \\ \text{as a variable} \end{matrix}$$

Substitute \textcircled{II} into \textcircled{IV}, multiply by γp

$$\textcircled{V} \quad \frac{Dp}{Dt} + \gamma p \frac{\partial y}{\partial x} = 0$$

This has removed ρ as a variable in two equations

Linearize \textcircled{I} and \textcircled{IV} as $u = \bar{u} + u'$; $p = \bar{p} + p'$; $e = \bar{e} + e'$

Expand total derivative $\frac{D(L)}{Dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial x} \frac{\partial x}{\partial t}$

$$\cancel{\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t}}^{\text{10}} + \cancel{\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{u} \frac{\partial u'}{\partial x}}^{\text{10}} + \cancel{u' \frac{\partial \bar{u}}{\partial x}}^{\text{10}} + \cancel{u' \frac{\partial u'}{\partial x}}^{\text{10}} \xrightarrow{\text{small, neglect}} = \frac{-1}{(\bar{e} + e')} \frac{\partial \bar{p}}{\partial x} - \frac{1}{(\bar{e} + e')} \frac{\partial p'}{\partial x}$$

Note from

$$\frac{1}{\bar{e}} \frac{1}{[1 + e'/\bar{e}]} \approx \frac{1}{\bar{e}} \left[1 - \frac{e'}{\bar{e}} \right] \xrightarrow{\text{small, neglect}} \frac{1}{\bar{e}}$$

$$\cancel{\frac{\partial \bar{p}}{\partial t} + \frac{\partial p'}{\partial t}}^{\text{10}} + \cancel{\bar{u} \frac{\partial \bar{p}}{\partial x} + \bar{u} \frac{\partial p'}{\partial x}}^{\text{10}} + \cancel{u' \frac{\partial \bar{p}}{\partial x}}^{\text{10}} + \cancel{u' \frac{\partial p'}{\partial x}}^{\text{10}} \xrightarrow{\text{small, neglect}} \\ + \gamma \bar{p} \cancel{\frac{\partial \bar{u}}{\partial x}}^{\text{10}} + \gamma \bar{p}' \cancel{\frac{\partial u'}{\partial x}}^{\text{10}} + \gamma \bar{p} \cancel{\frac{\partial u'}{\partial x}}^{\text{10}} + \gamma \bar{p}' \cancel{\frac{\partial p'}{\partial x}}^{\text{10}} \xrightarrow{\text{small, neglect}} = 0$$

The final linearized equations are

$$\textcircled{VI} \quad \frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} + \frac{1}{\bar{e}} \frac{\partial p'}{\partial x} = 0$$

$$\textcircled{VII} \quad \frac{\partial p'}{\partial t} + \bar{u} \frac{\partial p'}{\partial x} + \gamma \bar{p} \frac{\partial u'}{\partial x} = 0$$

The next step is to eliminate u' and obtain
a single equation for p' . Take $\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right)$ of (VII)

and substitute (VI)

$$\left(\frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x}\right) \text{(VII)} = \frac{\partial^2 p'}{\partial t^2} + \bar{u} \frac{\partial}{\partial x} \frac{\partial p'}{\partial t} + \bar{u} \frac{\partial}{\partial t} \frac{\partial p'}{\partial x}$$

$$+ \bar{u}^2 \frac{\partial^2 p'}{\partial x^2} + 8\bar{p} \frac{\partial}{\partial x} \frac{\partial u'}{\partial t} + \bar{u} 8\bar{p} \frac{\partial^2 u'}{\partial x^2} = 0$$

$$= -8\bar{p} \frac{\partial}{\partial x} \left(\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} \right)$$

since from (VI) $\frac{\partial u'}{\partial t} + \bar{u} \frac{\partial u'}{\partial x} = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x}$

Hence, the equation to solve is

$$\text{(VIII)} \quad \frac{\partial^2 p'}{\partial t^2} + 2\bar{u} \frac{\partial}{\partial x} \frac{\partial p'}{\partial t} + \bar{u}^2 \frac{\partial^2 p'}{\partial x^2} - 8\bar{p} \frac{\partial}{\partial x} \left(\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial x} \right) = 0$$

Assume a solution $p' = \hat{p} \exp \underbrace{[-i(kx - vt)]}_{\text{designate as } Q}$

$$\frac{\partial^2 \hat{p}}{\partial t^2} = i^2 v^2 \hat{p} \exp[Q]$$

$$2\bar{u} \frac{\partial^2 \hat{p}}{\partial x \partial t} = -2\bar{u} i^2 k v \hat{p} \exp[Q]$$

$$\bar{u}^2 \frac{\partial^2 \hat{p}}{\partial x^2} = \bar{u}^2 i^2 k^2 \hat{p} \exp[Q]$$

$$-\gamma \bar{p} \frac{\partial}{\partial x} \left(\frac{1}{\bar{p}} \frac{\partial \hat{p}}{\partial x} \right) = \frac{\gamma \bar{p}}{\bar{e}} i^2 k^2 \hat{p} \exp[Q]$$

Eliminate $-1, \hat{p}, \exp[Q]$, write VIII as

$$\underbrace{v^2 - 2\bar{u} k v + \bar{u}^2 k^2}_{(v - \bar{u} k)^2} = \frac{\gamma \bar{p}}{\bar{e}} k^2$$

Solve for v to get dispersion relationship

$$\textcircled{IX} \quad v = \bar{u} k \pm \sqrt{\frac{\gamma \bar{p}}{\bar{e}} k^2}$$

This can be written by pulling out k^2 from square root and $\bar{p} = \bar{e} R T$

$$\textcircled{X} \quad v = \bar{u} k \pm k \sqrt{\gamma R T}$$

The phase speed $c = \frac{v}{k}$ is

$$c = \bar{u} \pm \sqrt{\gamma R T}$$

\square \square

Doppler shift speed of sound

c is not a function of k . It is non-dispersive!

It is fortunate sound waves are non-dispersive.