Chapter 7

Spectral Methods

7.1 Introduction

The numerical integration methods discussed thus far are based on the discrete representation of the data on a grid or mesh of points covering the space over which a prediction of the variables is desired. Then a local time derivatives of the quantities to be predicted are determined by expressing the horizontal and vertical advection terms, sources etc., in finite difference form. Finally, the time extrapolation is achieved by one of many possible algorithms, for example leapfrog. The finite-difference technique has a number of associated problems such as truncation error, linear and nonlinear instability. Despite these difficulties, the finite-difference method has been the most practical method of producing forecasts numerically from the dynamical equations.

There is another approach called the *spectral* method which avoids some of the difficulties cited previously, in particular, nonlinear instability; however the method is less versatile and the required computations are comparatively time consuming. In a general sense, the mode of representation of data depends on the nature of the data and the shape of the region over which the representation is desired. An alternative to depiction on a mesh or grid of discrete points is a representation in the form of a series of orthogonal functions. This requires the determination of the coefficients of these functions, and the representation is said to be *spectral* representation or a series expansion in *wavenumber space*. When such functions are used, the *space derivatives can be evaluated analytically, eliminating the need for approximating them with finite-differences.*

As indicated earlier the choice of orthogonal functions depends in part on the geometry of the region to be represented, and for meteorological data a natural choice is a series of spherical harmonics. The first published work on the application of this technique to meteorological prediction is that of Silberman (1954). He considered the barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla(\zeta + f), \tag{7.1}$$

in spherical coordinates, where

$$\mathbf{e}_\theta \frac{1}{a} \frac{\partial}{\partial \theta} + e_\lambda \frac{1}{a \sin \theta} \frac{\partial}{\partial \lambda}$$

and $\mathbf{v} = v_\theta \mathbf{e}_\theta + v_\lambda \mathbf{e}_\lambda$

$$\frac{\partial \zeta}{\partial t} = -\frac{1}{a}\left(v_\theta \frac{\partial}{\partial \theta} + \frac{v_\lambda}{\sin\ \theta} \frac{\partial}{\partial \lambda} \right) (\zeta + 2\Omega\ \cos\ \theta\ ), \tag{7.2}$$

$$\zeta = \frac{1}{a \sin \theta} \left[\frac{\partial}{\partial \theta}(v_\lambda \sin \theta) - \frac{\partial v_\theta}{\partial \lambda} \right], \tag{7.3}$$

where a is the earth's radius, θ is the co-latitude, λ is the longitude, and v_λ and v_θ are the velocity components in the directions of increasing λ and θ. In terms of a stream function ψ, the velocity components from $\mathbf{v} = \mathbf{k} \times \nabla \psi$ are

$$v_\lambda = \frac{1}{a} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = -\frac{1}{a \sin \theta} \frac{\partial \psi}{\partial \lambda}$$

and the vorticity equation becomes with $\zeta = \nabla_s^2 \psi$

$$\nabla_s^2 \frac{\partial \psi}{\partial t} = \frac{1}{a^2 \sin \theta} \left(\frac{\partial \psi}{\partial \lambda} \frac{\partial}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial \lambda} \right) (\nabla_s^2 \psi + 2\Omega \cos \theta), \tag{7.4}$$

where

$$\nabla_s^2 = \frac{1}{a^2 \sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2}{\partial \lambda^2} \right].$$

The stream function is represented in terms of spherical harmonics which are the solutions Y_n^m of the equation

$$a^2 \nabla_s^2 Y_n^m + n(n+1) Y_n^m = 0. \tag{7.5}$$

These functions are expressible in the form

$$Y_n^m = e^{im\lambda} p_n^m. \tag{7.6}$$

If depend only on q if (7.6) is substituted into (7.5), the result is the ordinary differential equation (Exercise 7.1)

$$\frac{d^2 P_n^m}{d\theta^2} + \cot\theta \frac{dP_n^m}{d\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta} \right] P_n^m = 0. \tag{7.7}$$

This is a form of the Legendre equation, and its solutions P_n^m are known as *Legendre* functions of order m and degree n. The characteristic solutions of (7.7) are expressible as a set orthonormal functions such that

$$\int_0^\pi p_n^m p_s^m \sin\theta d\theta = \delta_{ns}, \tag{7.8}$$

where δ_{ns} is the Kronecker delta,

$$\delta_{ns} = 1 \quad \text{if} \quad n = s$$

and

$$\delta_{ns} = 0 \quad \text{if} \quad n \neq s.$$

The order m may take on negative values for which the legendre function is

$$P_n^{-m} = (-1)^m P_n^m.$$

For integer values of n and m, the Legendre functions are simply polynomials, the orders of which increase with n.

At any particular time t' the stream function may be expressed as the finite sum

$$\psi_{t=t'} = a^2\Omega \sum_{n=|m|}^{n'} \sum_{m=-m'}^{m'} K_n^m Y_n^m. \tag{7.9}$$

Since the series is finite, disturbances of sufficiently small-scale are not represented, which constitutes a truncation error; however, this is not necessarily undesirable. The harmonic coefficients K_n^{-m} are complex, and the condition for a real ψ is that

$$K_n^{-m} = (-1)^m K_n^{-m*}$$

where the * denotes the complex conjugate. The stream function tendency is given by

$$\left(\frac{\partial\psi}{\partial t}\right)_{t=t'} = a^2\Omega \sum_{n|m|}^{n''} \sum_{m=-m//}^{m''} \left(\frac{dK_n^m}{dt}\right)_{t=t'} Y_n^m. \tag{7.10}$$

When the expressions for ∂
partialpsi and ψ/t are substituted into the vorticity equation (7.4) and then

this equation is multiplied by $Y_n^{-m}\sin\theta$ and integrated from 0 to 2π with respect to λ and from 0 to π with respect to θ, the result is

$$\left(\frac{dK_n^m}{dt}\right)_{t=t'} = \frac{2i\Omega m K_n(t\prime)}{n(n+1)} \tag{7.11}$$
$$+\frac{i\Omega}{2}\sum_{s=|r|}^{n'}\sum_{r=-m'}^{m'}\sum_{k=|j|}^{n'}\sum_{j=-m'}^{m'} K_k^j(t')K_s^r(t')H_{kns}^{jmr},$$

where H_{kns}^{jmr} is zero unless $j+r=m$, in which case,

$$H_{kns}^{jmr} = \frac{s(s+1)-k(k+1)}{n(n+1)}\int_0^\pi P_n^m\left(jP_k^j\frac{dP_s^r}{d\theta} - r\frac{dP_k^j}{d\theta}P_s^r\right)d\theta. \tag{7.12}$$

The quantities H_{kns}^{jmr} are called *interaction coefficients*, which are zero unless

$$k+n+s = \text{odd integer} \quad \text{and} \quad |k-s| < n < k+s.$$

Also $m''2m'$ and $n'' \leq 2n'-1$. After the right side of the tendency equation (7.11) has been calculated, the future values of the expansion coefficients may be determined by extrapolating forward in time as in the finite-difference technique; for example, the leapfrog method:

$$K_n^m(t+\Delta t) = K_n^m(t-\Delta t) + 2\Delta t\frac{dK_n^m(t)}{dt}. \tag{7.13}$$

Robert (1966) pointed out that the components u and v of the wind field constitute pseudo-scalar fields on the globe, and as such are not well suited to representation in terms of scalar spectral expansions; he suggested that the variables

$$U = u\cos f \quad \text{and} \quad V = v\cos f, \tag{7.14}$$

where ϕ denotes latitude, would be more appropriate for global spectral representation.

The nondivergent flow field may be represented as usual in terms of a scalar stream function ψ as

$$\mathbf{V} = \mathbf{k}\times\nabla\psi.$$

Accordingly ζ is seen to be expressed as

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \nabla^2 \psi$$

and the quantities U and V as

$$U = -\frac{\cos\phi}{a}\frac{\partial\psi}{\partial\phi} \quad \text{and} \quad V = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} \tag{7.15}$$

The conservation of absolute vorticity may now be rewritten, with substitution of Eqs. (2) and (4), and an expansion into spherical polar coordinates as

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{1}{a\cos^2\phi}\left[\frac{\partial}{\partial\lambda}(U\nabla^2\psi) + \cos\phi\frac{\partial}{\partial\phi}(V\nabla^2\psi)\right] - 2\Omega\frac{V}{a}, \tag{7.16}$$

where ϕ, λ, a, and Ω denote, respectively, latitude, longitude, and the earth's radius and rotation rate. The linear equations (7.15) provide specification of the diagnostic quantities U and V in terms of the prognostic ψ.

The more usual form of the barotropic vorticity equation is seen on substitution of Eqs. (7.15) into (7.16) to be

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) = \frac{1}{a^2\cos\phi}\left[\frac{\partial\nabla^2\psi}{\partial\lambda}\frac{\partial\psi}{\partial\phi} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla^2\psi}{\partial\phi}\right] - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda}. \tag{7.17}$$

The calculation of the interaction coefficients is a lengthy task. An advantage of the method, however, is that nonlinear instability is avoided completely because all nonlinear interactions are computed analytically and all contributions to wave numbers outside the truncated series are automatically eliminated.

Robert (1966) proposed a modification to Silbermann's method for numerical integration of the primitive equations in which some simpler functions are substituted for the spherical harmonics. These functions are in fact the basic elements required to generate spherical harmonics, namely,

$$G_n^m(\alpha, \varphi) = e^{im\lambda}\cos^M\varphi\sin^n\varphi.$$

Here λ and φ are the longitude and latitude respectively, M is the absolute value of m and gives the number of waves along a latitude circle, and both M and n are either positive integers or zero.

7.2 An Example of the Spectral Method

As an illustration of the spectral method, we present a simple example due to Lorenz (1960). Consider the vorticity equation (7.1) applied to a plane region over which the stream function is doubly periodic, that is,

$$\psi\left(x+\frac{2\pi}{k},\, y+\frac{2\pi}{l},\, t\right)=\psi(x,y,t), \tag{7.18}$$

where k and l are constants. Thus the area is finite but unbounded, and in that respect it is analogous to the spherical earth. Note also that (7.1) applies with a constant coriolis parameter so that rotation is not excluded. Next assume that the stream function can be represented in terms of the characteristic solutions of the equation

$$\nabla^2\psi - c\psi = 0, \tag{7.19}$$

which is the analogue to (7.5). The solutions are trigonometric functions; thus ψ is expressible as a double Fourier series, which for convenience may be written as

$$\psi=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}-\frac{1}{m^2k^2+n^2l^2}\left[A_{mn}\cos(mkx+nly)+B_{mn}\sin(mkx+nly)\right]. \tag{7.20}$$

The coefficients are as yet unknown functions of time, except for the initial values which are assumed known. It is apparent from (7.16) that the characteristic values of (7.19) are $c_{mn}=-(m^2k^2+n^2l^2)$.

For actual prediction purposes a finite series of the form (7.20) is used and the time derivatives of the coefficients must be determined from the vorticity equation. Consider a simple case where m and n take on only the values of $0, 1, -1$.

Then after combining like terms, (7.20) is expressible in the form

$$\begin{aligned}\psi &= -\tfrac{A_{10}}{k^2}\cos kx-\tfrac{A_{01}}{l^2}\cos ly-\tfrac{A_{11}}{k^2+l^2}\cos(kx+ly)\\ &-\tfrac{A_{1-1}}{k^2+l^2}\cos(kx-ly)-\tfrac{B_{10}}{k^2}\sin kx-\tfrac{B_{01}}{l^2}\sin ly\\ &-\tfrac{B_{11}}{k^2+l^2}\sin(kx+ly)-\tfrac{B_{1-1}}{k^2+l^2}\sin(kx-ly).\end{aligned}$$

It turns out that, if the B's are zero initially, they will remain so; also if $A_{1-1}-A_{11}$ initially, it will remain so. With these assumptions, Lorenz obtains the 'maximum specification' of the stream function for use with (7.1), namely,

$$\psi=-\frac{A}{l^2}\cos ly-\frac{F}{k^2}\cos kx-\frac{2G}{k^2+l^2}\sin ly\sin kx. \tag{7.21}$$

Substituting this streamfunction into the vorticity equation (7.1), utilizing trigonometric identities, and finally equating coefficients of like terms leads to the following differential equations for the coefficients:

$$
\begin{aligned}
\frac{dA}{dt} &= -\left(\frac{1}{k^2} - \frac{1}{k^2+l^2}\right) klFG \equiv K_1 FG, \\
\frac{dF}{dt} &= \left(\frac{1}{l^2} - \frac{1}{k^2+l^2}\right) klAG \equiv K_2 AG, \\
\frac{dG}{dt} &= -\frac{1}{2}\left(\frac{1}{l^2} - \frac{1}{k^2}\right) klAF \equiv K_3 AF.
\end{aligned} \tag{7.22}
$$

Note that the interaction coefficients K_1, K_2, and K_3 are constants and hence, remain the same throughout the period of integration. The set (7.22), which is analogous to (12.53), can be solved analytically; however when the spectral technique is applied, say to a hemisphere with real data, the resulting system of equations is much more complex and must be solved by numerical methods. If the leapfrog scheme is used here, the numerical integration of (7.22) would be analogous to (7.13), that is,

$$
A^{\tau+1} = A^{\tau-1} + 2\Delta t K_1 F^{\tau} G^{\tau}, \tag{7.23}
$$

etc., where the superscript denotes the time step. As usual, a forward step must be used for the first time step from the initial values of the coefficients A_0, F_0, and G_0. To avoid linear instability, Δt must be a fairly small fraction of the period of the most rapidly oscillating variable. Equation (7.23) and similar ones for F and G permit the calculation of future values of the coefficient of the series (7.21); thus the prediction of a stream function is achieved.