Splines

1) A spline is a series of piecewise equations
2) Each derived equation is used to interpolate between two data points
3) The two data points are known as knots
4) In many cases, the equations are third-order polynomials, but they can be other nth-order polynomials or other functions fitting an equation between two data points

For simplicity, we will show examples with a linear spline and a quadratic spline.

We will then discuss cubic splines

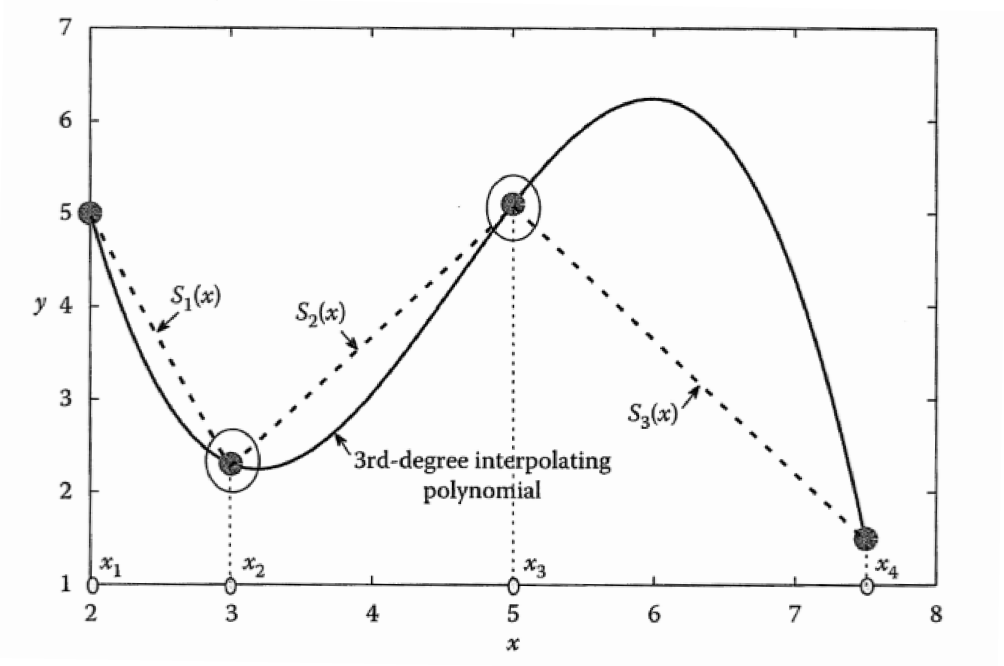
Linear spline

1) This is just linear interpolation
2) Simplest example to show formulation of splines
3) A possible third-order polynomial curve is also shown

$$S_1(x) = \frac{x - x_2}{x_1 - x_2} y_1 + \frac{x - x_1}{x_2 - x_1} y_2, \quad x_1 \le x \le x_2$$

$$S_2(x) = \frac{x - x_3}{x_2 - x_3} y_2 + \frac{x - x_2}{x_3 - x_2} y_3, \quad x_2 \le x \le x_3$$

$$S_3(x) = \frac{x - x_4}{x_3 - x_4} y_3 + \frac{x - x_3}{x_4 - x_3} y_4, \quad x_3 \le x \le x_4$$



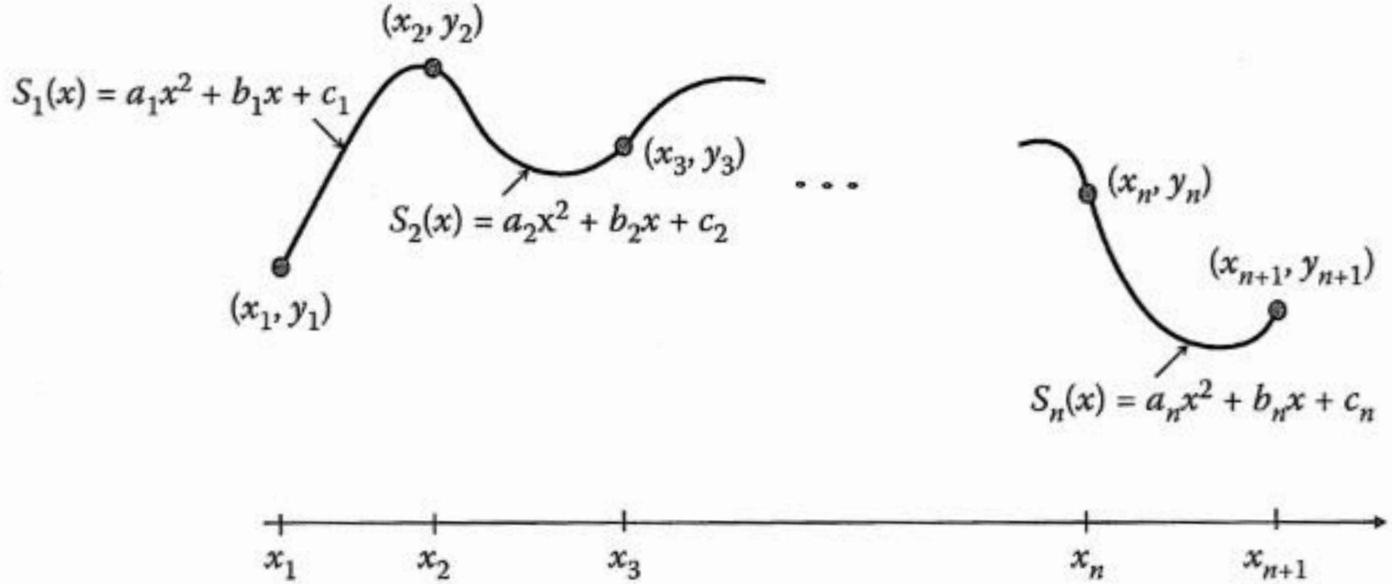
Quadratic spline

If one has n+1 x,y data points, a quadratic spline is of the form

$$S_i(x) = a_i\, x^2 + b_i\, x + c_i, \quad i = 1, 2, \dots, n$$

The seek to form a closed set of equations to solve for each piecewise polynomial. The first polynomial $S_1(x)$ must go through (x_1,y_1) and the last polynomial $S_n(x)$ must go through (x_{n+1},y_{n+1})

$$\left.\begin{aligned} S_1(x_1) &= y_1 \\ S_n(x_{n+1}) &= y_{n+1} \end{aligned}\right\}_{\text{2equations}}$$



At the interior data points ("knots"), Each polynomial must go through, and adjacent polynomial must agree at their respective data points.

$$S_i(x_{i+1}) = y_{i+1}, \qquad i = 1, 2, \ldots, n-1$$
$$S_i(x_i) = y_i, \qquad i = 2, 3, \ldots, n$$

So that:

$$a_i x_{i+1}^2 + b_i x_{i+1} + c_i = y_{i+1}, \qquad i = 1, 2, \ldots, n-1$$
$$a_i x_i^2 + b_i x_i + c_i = y_i, \qquad i = 2, 3, \ldots, n$$

To make function smooth, also set the first derivatives at interior points equal

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), \quad i = 1, 2, \ldots, n-1$$

So that:

$$2a_i x_{i+1} + b_i = 2a_{i+1} x_{i+1} + b_{i+1}, \quad i = 1, 2, \ldots, n-1$$

We need one more equation to achieve a closed solution. Let's set the second derivative at x_1 to zero, and noting that $S_1''(x_1) = 2a_1 = 0$

$$a_1 = 0$$

Quadratic spline example

x_i	y_i
2	5
3	2.3
5	5.1
7.5	1.5

At (x_1,y_1) and (x_4,y_4)

$$a_1(2)^2 + b_1(2) + c_1 = 5$$
$$a_3(7.5)^2 + b_3(7.5) + c_3 = 1.5$$

At x_1

$$a_1 = 0$$

At the interior data points

$$a_1(3)^2 + b_1(3) + c_1 = 2.3$$
$$a_2(5)^2 + b_2(5) + c_2 = 5.1$$
$$a_2(3)^2 + b_2(3) + c_2 = 2.3$$
$$a_3(5)^2 + b_3(5) + c_3 = 5.1$$

Set the first derivatives at interior points equal

$$2a_1(3) + b_1 = 2a_2(3) + b_2$$
$$2a_2(5) + b_2 = 2a_3(5) + b_3$$

Quadratic spline example

Writing these equations in matrix form

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 56.25 & 7.5 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 25 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 & 5 & 1 \\ 1 & 0 & -6 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 1 & 0 & -10 & -1 & 0 \end{bmatrix} \begin{Bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 1.5 \\ 2.3 \\ 5.1 \\ 2.3 \\ 5.1 \\ 0 \\ 0 \end{Bmatrix}$$

Which can be solved to obtain

$$\boxed{a_1 = 0} \quad a_2 = 2.05 \quad a_3 = -2.776$$
$$b_1 = -2.7 \quad b_2 = -15 \quad b_3 = 33.26$$
$$c_1 = 10.4 \quad c_2 = 28.85 \quad c_3 = -91.8$$

And the three spline equations are:

$$S_1(x) = -2.7x + 10.4, \quad 2 \le x \le 3$$
$$S_2(x) = 2.05x^2 - 15x + 28.85, \quad 3 \le x \le 5$$
$$S_3(x) = -2.776x^2 + 33.26x - 91.8, \quad 5 \le x \le 7.5$$

Cubic spline

The methodology is the similar as the quadratic spline, except it's a third order polynomial with n powers to $(x-x_i)$

$$S_i(x) = a_i\,(x - x_i)^3 + b_i\,(x - x_i)^2 + c_i\,(x - x_i) + d_i, \quad i = 1, 2, \ldots, n$$

With the following same conditions as before:

$$S_1(x_1) = y_1, \quad S_n(x_{n+1}) = y_{n+1}$$

$$S_{i+1}(x_{i+1}) = S_i(x_{i+1}), \quad i = 1, 2, \ldots, n-1$$

$$S_i(x_i) = y_i, \quad i = 2, 3, \ldots, n$$

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), \quad i = 1, 2, \ldots, n-1$$

But setting the 2nd derivatives equal to each other

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}), \quad i = 1, 2, \ldots, n-1$$

The formulas for above are straightforward algebra but tedious. Details are on the website.

The final step is setting boundary conditions. There are several options

Some types of cubic spline boundary conditions

- $S_1 = S_n = 0$, which assumes the end cubics approaches linearity at their extremities, called a *natural spline*
- $S_1 = S_2,$ and $S_n = S_{n+1}$, which assumes the end cubics approach parabolas at their extremities
- Take S_1 as a linear extrapolation from S_2 and S_3, and S_n as a linear extrapolation from S_{n-1} and S_{n-2}
- ${S_1}' = constant\ value$, and ${S_n}' = another\ constant\ value$, known as *clamped boundary conditions.* This forces the slope at each end to assume certain values.
- ${S_1}'' = {S_n}'' = 0$, known as *free boundary conditions*
- $S_1 = S_n$, *periodic boundary condition*. Only use for a periodic function

Comparison of quadratic spline and cubic spline with clamped boundary conditions