3.1 Classification of partial differential equations (PDEs)

3.1.1 Reminder about PDEs

Second order linear PDE

$$\alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial x \partial y} + \gamma \frac{\partial^2 u}{\partial y^2} + 2\delta \frac{\partial u}{\partial x} + 2\varepsilon \frac{\partial u}{\partial x} + \varphi u = 0$$

Second order linear partial differential equations are classified into three types depending on the sign of $\beta^2 - \alpha\gamma$ (e.g., Courant and Hilbert, 1962). Equations are hyperbolic, parabolic or elliptic if the sign is positive, zero, or negative, respectively. The simplest (canonical) examples of these equations are

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ Wave equation (hyperbolic).
Example: vibrating string.

(b) $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$ Diffusion equation (parabolic).
Example: heated rod.

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ (or\ f(x, y))$ Laplace's or Poisson's equations (elliptic).
Examples: steady state temperature of a plate, streamfunction/vorticity relationship.

The behavior of the solutions, the proper initial and/or boundary conditions, and the numerical methods that can be used to find the solutions *depend essentially on the type* *of PDE* that we are dealing with. Although nonlinear multidimensional PDEs cannot in general be reduced to these canonical forms, we need to study these prototypes of the PDEs to develop an understanding of their properties, and then apply similar methods to the more complicated NWP equations.

(d) $\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x}$ Advection equation, with solution $u(x, t) = u(x - ct, 0)$.

The advection equation is a first order PDE, but it can also be classified as a hyperbolic, since its solutions satisfy the wave equation (a), and the latter is usually written as the system

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{A}\frac{\partial \mathbf{u}}{\partial x}$$

where

$$\mathbf{u} = \begin{pmatrix} \frac{\partial u}{\partial t} \\ c\frac{\partial u}{\partial x} \end{pmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix} \text{ or an equivalent transformation}$$