Vectorial Form of Momentum Eq

Let $\vec{A}$ be an arbitrary vector.

In an absolute (inertial) frame of reference:

(1) $$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

while in a relative (noninertial) frame of reference which rotates with an angular velocity $\vec{\Omega}$

(2) $$\vec{A} = A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'$$

The time rate of change of $\vec{A}$ in an inertial frame is

(3) $$\frac{d_a \vec{A}}{dt} = \frac{d}{dt}\left(A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'\right)$$

$$= \frac{dA_x'}{dt} \hat{i}' + \frac{dA_y'}{dt} \hat{j}' + \frac{dA_z'}{dt} \hat{k}'$$

$$+ A_x' \frac{d\hat{i}'}{dt} + A_y' \frac{d\hat{j}'}{dt} + A_z' \frac{d\hat{k}'}{dt}$$

since both the components of $\vec{A}$ and the unit vectors change with time.

However, for an observer in a non inertial reference frame, the unit vectors are fixed in length and direction. Therefore

$$(4)\quad \frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{i} + \frac{dA_y}{dt}\hat{j} + \frac{dA_z}{dt}\hat{k}$$

However since $\vec{A}$ is the same in both coordinate systems

$$(5)\quad \frac{d\vec{A}}{dt} = \frac{dA_x'}{dt}\hat{i}' + \frac{dA_y'}{dt}\hat{j}' + \frac{dA_z'}{dt}\hat{k}'$$

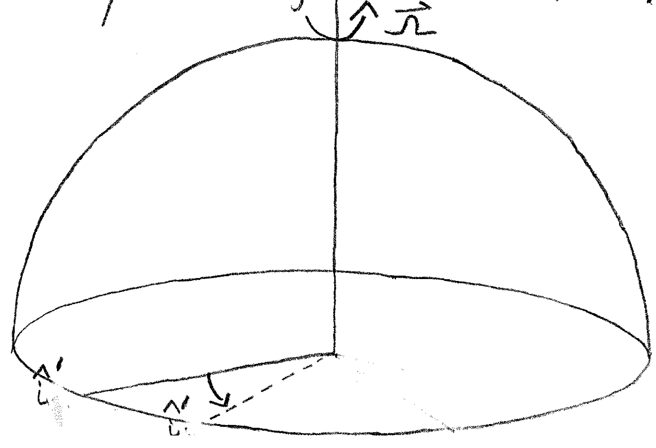
Upon substituting (5) into (3)

$$(6)\quad \frac{d_a\vec{A}}{dt} = \frac{d\vec{A}}{dt} + A_x'\frac{d\hat{i}'}{dt} + A_y'\frac{d\hat{j}'}{dt} + A_z'\frac{d\hat{k}'}{dt}$$

$\frac{d\hat{i}'}{dt}$ is the velocity owing to its rotation.

Therefore

$$\frac{d\hat{i}'}{dt} = \vec{\Omega}\times\hat{i}'$$



Similarly,

$$\frac{D\hat{j}'}{Dt} = \vec{\Omega} \times \hat{j}' \quad ; \quad \frac{D\hat{k}'}{Dt} = \vec{\Omega} \times \hat{k}'$$

Which gives

$$A_x' \frac{d\hat{i}'}{dt} + A_y' \frac{d\hat{j}'}{dt} + A_z' \frac{d\hat{k}'}{dt} \tag{7}$$

$$= A_x' \vec{\Omega} \times \hat{i}' + A_y' \vec{\Omega} \times \hat{j}' + A_z' \vec{\Omega} \times \hat{k}'$$

$$= \vec{\Omega} \times (A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}')$$

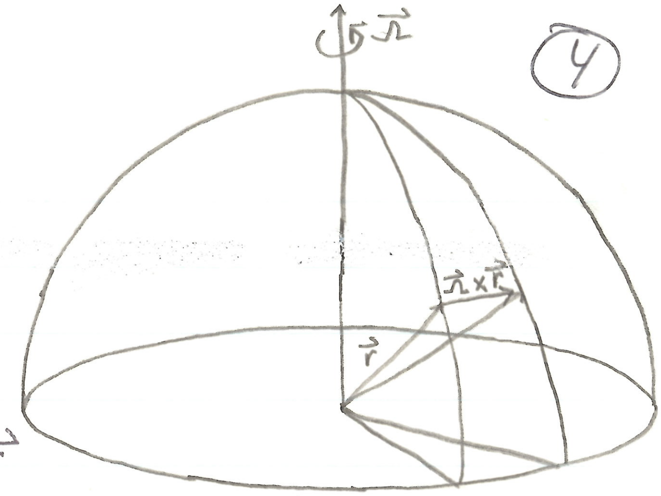
$$= \vec{\Omega} \times \vec{A}$$

Substitute 7 into 6

$$\boxed{\frac{d_a \vec{A}}{dt} = \frac{d\vec{A}}{dt} + \vec{\Omega} \times \vec{A}} \tag{8}$$

Thus, the rates of time of the same vector $\vec{A}$ are perceived differently in the rotating and nonrotating frames. The term $\vec{\Omega} \times \vec{A}$ must be added to $\frac{d\vec{A}}{dt}$ (in rotating frame) to accurately describe $\frac{d_a \vec{A}}{dt}$ (in nonrotating frame).

Define $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
called the "position vector"



Now let $\vec{A} = \vec{r}$
According to (8)

(9) $$\frac{d_a\vec{r}}{dt} = \frac{d\vec{r}}{dt} + \vec{\Omega}\times\vec{r}$$

So that the velocity seen in the nonrotating frame ($\vec{V}_a$) is equal to the velocity observed in the rotating (non inertial) frame augmented by the velocity imparted by solid-body rotation ($\vec{\Omega}\times\vec{r}$) of the earth. $\frac{d\vec{r}}{dt}$ is sometimes called "relative velocity." (9) may be written as

(10) $$\vec{V}_a = \vec{V} + \vec{\Omega}\times\vec{r}$$

Now let $\vec{A} = \vec{V}_a$, and substitue (10) into (8)

(11) $$\frac{d_a\vec{V}_a}{dt} = \frac{d(\vec{V} + \vec{\Omega}\times\vec{r})}{dt} + \vec{\Omega}\times(\vec{V} + \vec{\Omega}\times\vec{r})$$

Expand (11)

(12) $$\frac{d_a\vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + \frac{d\vec{\Omega}}{dt}\times\vec{r} + \vec{\Omega}\times\frac{d\vec{r}}{dt}$$

$$+ \vec{\Omega}\times\vec{V} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

Since $\frac{d\vec{\Omega}}{dt} = 0$, $\frac{d\vec{r}}{dt} = \vec{V}$, and combining terms

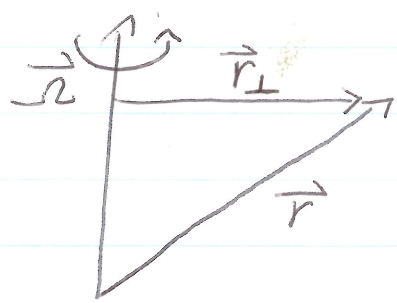
$$(13) \quad \boxed{\frac{d_a \vec{V}_a}{dt} = \frac{d\vec{V}}{dt} + 2\vec{\Omega} \times \vec{V} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})}$$

The 2nd term is the Coriolis force.
The 3rd term is the centrifugal force.

Aside

The centrifugal force may be rewritten in terms of $\vec{r}_\perp$, the perpendicular distance vector from the rotation axis to the position of the fluid element at $\vec{r}$.

* Holton uses $\vec{R}$ instead of $\vec{r}_\perp$



Since $\vec{\Omega} \times \vec{r} = \vec{\Omega} \times \vec{r}_\perp$, we have

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -|\vec{\Omega}|^2 \vec{r}_\perp$$

with the aid of the identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

In the absolute observer's frame of reference, the sum of forces:

$$\frac{d_a \vec{V}_a}{dt} = \Sigma \frac{\vec{F}}{m}$$

are gravitation, the pressure gradient force, and friction

(14) $$\Sigma \frac{\vec{F}}{m} = \vec{G} - \frac{1}{\rho} \nabla p + \vec{F_r}$$

Substituting (14) into (13) and solving for $\frac{d\vec{V}}{dt}$

(15) $$\frac{d\vec{V}}{dt} = \vec{G} - \frac{1}{\rho} \nabla p + \vec{F_r} - 2\vec{\Omega} \times \vec{V} - \vec{\Omega} \times (\vec{\Omega} \times \vec{V})$$

We usually lump $\vec{G}$ and $-\vec{\Omega} \times (\vec{\Omega} \times \vec{V})$ together

* effective gravity (16) $$\vec{g} = \vec{G} - \vec{\Omega} \times (\vec{\Omega} \times \vec{V})$$

Therefore, Newton's second law of motion relative to a rotating (noninertial) coordinate frame is

(17) $$\boxed{\frac{d\vec{V}}{dt} = \vec{g} - \frac{1}{\rho} \nabla p + \vec{F_r} - 2\vec{\Omega} \times \vec{V}}$$