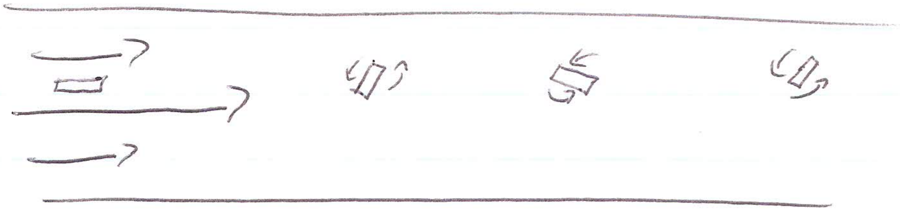
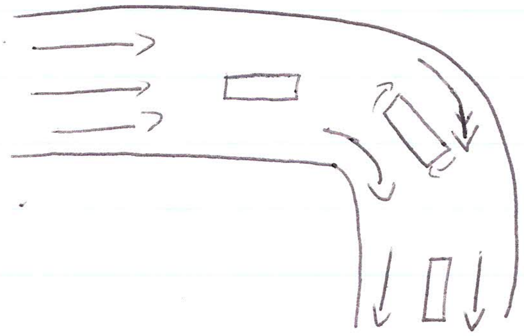
Vorticity

It should come as no surprise that atmospheric motion has a significant amount of "spin" associated with it (even in apparent straight line flow). By studying background rotation, we can learn a great deal about atmospheric phenomena.

Consider a stick floating in a stream:



or



Both give evidence that parcels have a spin imposed on their translational motion.

Background rotation is observed at all scales: a leaf in the wind, dust devils, satellite photos of low, hurricanes.

Fluid dynamicists and atmospheric scientists have found it useful to study the spin of a fluid as a separate variable derived from the motion field by differentiation. It is called <u>vorticity</u>. We will attempt to show how vorticity quantifies the spin of a parcel.

3-D vorticity $\vec{\omega}$

3-D Vorticity is defined as the curl of the velocity field:

(1) $$\vec{\omega} = \nabla \times \vec{V} = \hat{i}\underbrace{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)}_{\xi} + \hat{j}\underbrace{\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)}_{\eta} + \hat{k}\underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\zeta}$$

The components of the vorticity ξ, η, and ζ are measures of the spin about the x, y, and z axes. Most meteorologists are concerned about only the vertical component ζ on the synoptic scale, although the other components can be important (i.e., in tornadogenesis).

<u>Vertical component of vorticity</u>

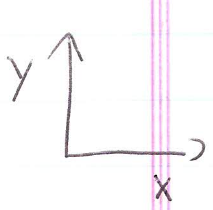
By definition, the vertical component of vorticity is

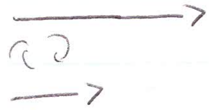
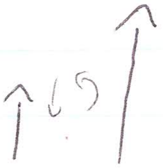
(2) $$\zeta = \hat{k} \cdot \nabla \times \vec{V} = \hat{k} \cdot \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \hat{i}u + \hat{j}v + \hat{k}w$$

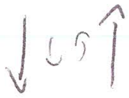
or

(3) $$\boxed{\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}$$

$\zeta > 0$ in cyclonic flow, $\zeta < 0$ in anticyclonic flow.
For example:



 $-\frac{\partial u}{\partial y} < 0$,  $\frac{\partial v}{\partial x} > 0$,

 $\frac{\partial v}{\partial x} > 0$  $-\frac{\partial u}{\partial y} > 0$  $-\frac{\partial u}{\partial y} < 0$

ζ in natural coordinates

Two different derivations

Recall that, in natural coordinates, $\vec{V} = |\vec{V}|\hat{\tau}$. Thus

$$\zeta = \hat{k} \cdot \nabla_H \times \vec{V} = \hat{k} \cdot \left[\left(\hat{\tau}\frac{\partial}{\partial s} + \hat{n}\frac{\partial}{\partial n} \right) \times \left(\hat{\tau}|\vec{V}| \right) \right]$$

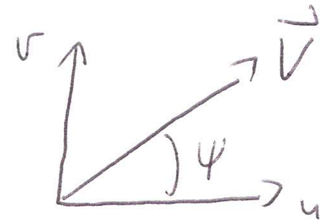
$$= \hat{k} \cdot \left[\underbrace{\cancel{(\hat{\tau} \times \hat{\tau})}}_{0} \frac{\partial |\vec{V}|}{\partial s} + |\vec{V}| \Big(\underbrace{\hat{\tau} \times \frac{\partial \hat{\tau}}{\partial s}}_{K\hat{k}} \Big) + \underbrace{(\hat{n} \times \hat{\tau})}_{-\hat{k}} \frac{\partial |\vec{V}|}{\partial n} + |\vec{V}| \Big(\underbrace{\cancel{\vec{n} \times \frac{\partial \hat{\tau}}{\partial n}}}_{0} \Big) \right]$$

(4a) $$\boxed{\zeta = |\vec{V}| K - \frac{\partial |\vec{V}|}{\partial n} = \frac{|\vec{V}|}{R} - \frac{\partial |\vec{V}|}{\partial n}}$$

Alternative approach:

streamlines in horizontally varying windspeed field

The vector $\vec{V}$ may be broken into its components u and v



$$u = |\vec{V}| \cos\psi$$

$$v = |\vec{V}| \sin\psi$$

Therefore:

$$\frac{\partial v}{\partial x} = \sin\psi \frac{\partial |\vec{V}|}{\partial x} + |\vec{V}| \cos\psi \frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial y} = \cos\psi \frac{\partial |\vec{V}|}{\partial y} - |\vec{V}| \sin\psi \frac{\partial \psi}{\partial y}$$

Now rotate x, y axes so that $\psi \to 0$. Then $\cos\psi \to 1$ and $\sin\psi \to 0$. Also, $x \to s$ and $y \to n$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = |\vec{V}| \underbrace{\frac{\partial \psi}{\partial s}}_{K = \frac{1}{R}} - \frac{\partial |\vec{V}|}{\partial n}$$

4b $$\boxed{\zeta = \frac{|\vec{V}|}{R} - \frac{\partial |\vec{V}|}{\partial n}}$$

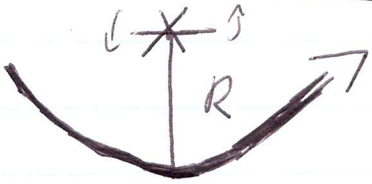
Interpretation: Net vertical vorticity is the sum of two parts: 1) the turning of the wind along a streamline $|\vec{V}|/R$, called <u>curvature vorticity</u>; 2) the rate of change of wind speed normal to the direction of flow, $-\partial|\vec{V}|/\partial n$, called <u>shear vorticity</u>,

$$\zeta = \text{curvature vorticity} + \text{shear vorticity}$$

Examples:

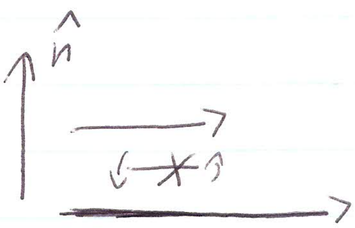
<u>Cyclonic curvature vorticity</u>

$$\frac{|\vec{V}|}{R} > 0$$



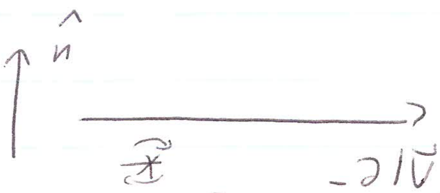
<u>Anticyclonic curvature vorticity</u>

<u>Cyclonic shear vorticity</u>



$$-\frac{\partial|\vec{V}|}{\partial n} > 0$$

<u>Anticyclonic shear vorticity</u>



$$-\frac{\partial|\vec{V}|}{\partial n} < 0$$

Curvature and shear vorticity tend to <u>not</u> <u>cancel</u>, hence it is a useful quantity in meteorology

$\zeta \sim$ Order of $10^{-5}\ s^{-1}$

"Earth vorticity"

Since the earth imparts a background rotation (as seen in the figure), it also affects the vorticity of an air parcel. Since the source of rotation is the earth, not the wind field, this is called "earth vorticity." It turns out earth vorticity is equal to the Coriolis parameter "f"

$$f = 2\Omega \sin\phi$$

and is always cyclonic.

Proof:

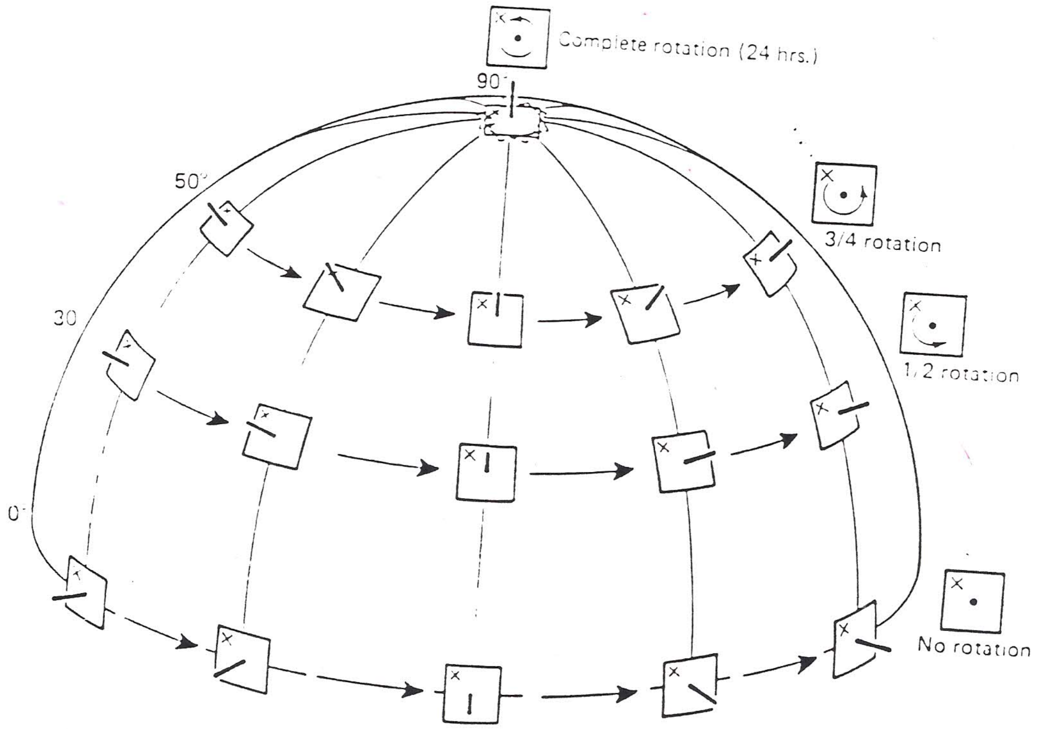
The velocity of an air parcel due to the earth's motion is

$$\vec{V}_e = \vec{\Omega} \times \vec{r}$$

where $\vec{r} \equiv$ position vector from the center of the earth to an air parcel, and $\vec{\Omega} = \Omega\cos\phi\,\hat{j} + \Omega\sin\phi\,\hat{k}$

$$\zeta_e = \hat{k}\cdot\nabla\times\vec{V}_e = \hat{k}\cdot\underbrace{\nabla\times(\vec{\Omega}\times\vec{r})}_{2\vec{\Omega}} = 2\Omega\sin\phi$$

(5) $$\boxed{\zeta_e = 2\Omega\sin\phi = f}$$



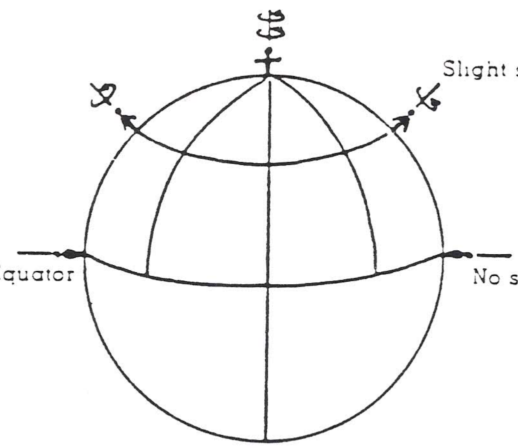


Fig. 17.13 Due to the rotation of the earth, the rate of spin of observers about their vertical axes increases from zero at the equator to a maximum at the poles.

Absolute vorticity

3D version ($\vec{\omega}_a$) :

(6) $$\boxed{\vec{\omega}_a = \nabla \times \vec{V}_a}$$

$$\underbrace{\vec{V}_a}_{\text{absolute velocity}} = \underbrace{\vec{V}}_{\text{relative velocity}} + \underbrace{\vec{V}_e}_{\text{velocity of parcel due to earth rotation}}$$

(7) $$\boxed{\vec{\omega}_a = \underbrace{\nabla \times \vec{V}}_{\vec{\omega}} + \underbrace{\nabla \times \vec{V}_e}_{\vec{\omega}_e = 2\vec{\Omega}}}$$

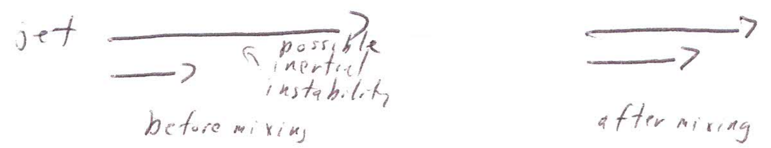
Vertical component version (ζ_a) :

(8) $$\boxed{\zeta_a = \underbrace{\zeta}_{\text{relative vorticity}} + \underbrace{f}_{\text{earth vorticity}}}$$

$\zeta_a > 0$ almost always, since $f > |\zeta|$.

One exception : anticyclonic side of polar jet. Under some conditions, if the anticyclonic shear became too large, unstable motions would occur which would mix the fluid laterally and reduces the shear until $\zeta_a > 0$ again. This is called inertial instability. Might cause turbulence around jets.

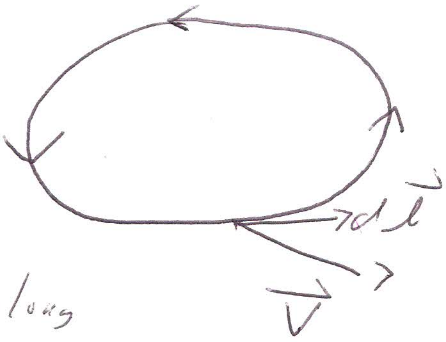
Also called dynamic instability.



Other versions of vorticity

Circulation is defined as the line integral about the component of $\vec{V}$ tangent to a closed curve is

$$C = \oint \vec{V} \cdot d\vec{l}$$



where $\vec{l}$ is the incremental length along a closed circle.

Circulation, which is a scalar integral quantity, is a macroscopic measure of rotation for a finite area.

Vorticity is a vector field that gives a microscopic measure of the rotation at any point.

$$\zeta \equiv \lim_{area \to 0} \frac{\left(\oint \vec{V} \cdot d\vec{l}\right)}{area}$$