Fourier Series (also called Harmonic Analysis)

A wave in the x direction may be decomposed into trigonometric components of the form:

$$\Psi(x) = \sum_{n=1}^{\infty} (A_n \sin k_n x + B_n \cos k_n x)$$

For a domain length of L where $k = \frac{2\pi}{L}$ for n waves

In practice, n is truncated to a reasonable wave number

$$\Psi(x) = \sum_{n=1}^{\infty} \left(A_n \sin\frac{2\pi n x}{L} + B_n \cos\frac{2\pi n x}{L}\right)$$

The Fourier coefficients A_n and B_n are expressed as

The number of coefficients which can be retrieved is half of i_{last}

$$A_n = \frac{2}{L} \sum_{i=1}^{i_{last}} \Psi_i(x) \sin\frac{2\pi n x}{L} \Delta x$$

$$B_n = \frac{2}{L} \sum_{i=1}^{i_{last}} \Psi_i(x) \cos\frac{2\pi n x}{L} \Delta x$$

where at i_{last} L is the length of the domain

In calculus notation, this is written as

$$A_n = \frac{2}{L}\int_0^L \Psi(x) \sin\frac{2\pi n x}{L} dx$$

$$B_n = \frac{2}{L}\int_0^L \Psi(x) \cos\frac{2\pi n x}{L} dx$$

This allows one to retrieve A and B for each frequency!

The Fourier coefficient C_n is the amplitude of the combined $A_n \sin\frac{2\pi nx}{L}$ and $B_n \sin\frac{2\pi nx}{L}$ terms for each n

$$C_n = \sqrt{A_n^2 + B_n^2}$$

with a phase shift ϕ_n computed by $\tan\phi_n = \frac{B_n}{A_n}$.
The proof is below.

The signal power for each n is

$$P_n = \frac{1}{2} C_n^2$$

The variance explained by each harmonic is

$$R_n^2 = \frac{i_{last}\, P_n}{(i_{last}-1)\, s^2} \times 100\%$$

where i_{last} is the last i index at L in the discretized domain.

s^2 is the sample variance

$$s^2 = \frac{\sum_{i=1}^{i_{last}} (\psi_i - \bar{\psi})^2}{i_{last} - 1}$$

The ratio of $\frac{i_{last}}{i_{last}-1}$ is a statistical artifact compensating for the fact this is a limited data sample, not a statistical population. This correction factor is ignored in most books

Proof for C_n

If $\tan\phi = \frac{B}{A}$, it can be shown from Pythagorean's theorem:

$$\sin\phi = \frac{B}{\sqrt{A^2+B^2}} \quad \text{and} \quad \cos\phi = \frac{A}{\sqrt{A^2+B^2}}$$

Consider:

$$\begin{aligned}
\Psi(x) &= A\sin x + B\cos x \\
&= \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\sin x + \frac{B}{\sqrt{A^2+B^2}}\cos x\right] \\
&= \underbrace{\sqrt{A^2+B^2}}_{C}\Big[\underbrace{\sin x\cos\phi + \cos x\sin\phi}_{\substack{\text{trig identity!} \\ \sin(x+\phi)}}\Big]
\end{aligned}$$

Hence, $\Psi(x)$ can be rewritten:

$$\Psi(x) = C\sin(x+\phi)$$