Classical Rossby wave dispersion equation

Assume a barotropic, non-divergent atmosphere in one dimension (along x axis)

$$\frac{D(\zeta+f)}{Dt} = 0$$

Expand total derivative

$$\frac{\partial}{\partial t}(\zeta+f) + u\frac{\partial}{\partial x}(\zeta+f) + v\frac{\partial}{\partial y}(\zeta+f) = 0$$

Make the "Beta-plane" approximation $f = f_0 + \beta y$ where $\beta = \frac{\partial f}{\partial y}$
Also note that $\frac{\partial f}{\partial t} = 0$ and $\frac{\partial f}{\partial x} = 0$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + v\beta = 0$$

Linearize as $\underbrace{u = \bar{u}}_{\text{background mean zonal wind}} + u'$; $v = v'$; $\zeta = \zeta'$

Variations are only along x except for β, v'

$$\frac{\partial \zeta'}{\partial t} + \bar{u}\frac{\partial \zeta'}{\partial x} + v'\beta = 0$$

Note the $v' = \frac{\partial \psi}{\partial x}$, $\zeta' = \frac{\partial v'}{\partial x} - \cancel{\frac{\partial u'}{\partial y}}^{\to 0} = \frac{\partial^2 \psi'}{\partial x^2}$

no variation of u' along y

(I) $$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right) + \frac{\partial \psi}{\partial x}\beta = 0}$$

Assume a solution of $\psi = \hat{\psi}\exp[\underbrace{i(kx - \nu t)}_{\text{write as } Q \text{ to shorten notation}}]$

Since $\boxed{\frac{\partial \psi}{\partial x} = ik\hat{\psi}\exp[Q]}$ (II) and $\frac{\partial^2 \psi}{\partial x^2} = \underbrace{i^2}_{-1} k^2 \hat{\psi}\exp[Q]$

and then $\longrightarrow$ $\boxed{\frac{\partial}{\partial t}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = i\nu k^2 \hat{\psi}\exp[Q]}$ (III)

(Note that another negative from $\frac{\partial}{\partial t}$ of $\exp[i(kx-\nu t)]$ creates another negative sign

and $\longrightarrow$ $\boxed{\frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right) = -ik^3 \hat{\psi}\exp[Q]}$ (IV)

Plug (II), (III) and (IV) into (I)

$$i\nu k^2 \hat{\psi}\exp[Q] - \bar{u} i k^3 \hat{\psi}\exp[Q] + ik\hat{\psi}\exp[Q]\beta = 0$$

Note that i cancels out, a k cancels out, and $\hat{\psi}\exp[Q]$ cancels out