Normal modes

- **modes** = harmonic (sinusoidal) motions.
- **normal** = independent of each other.

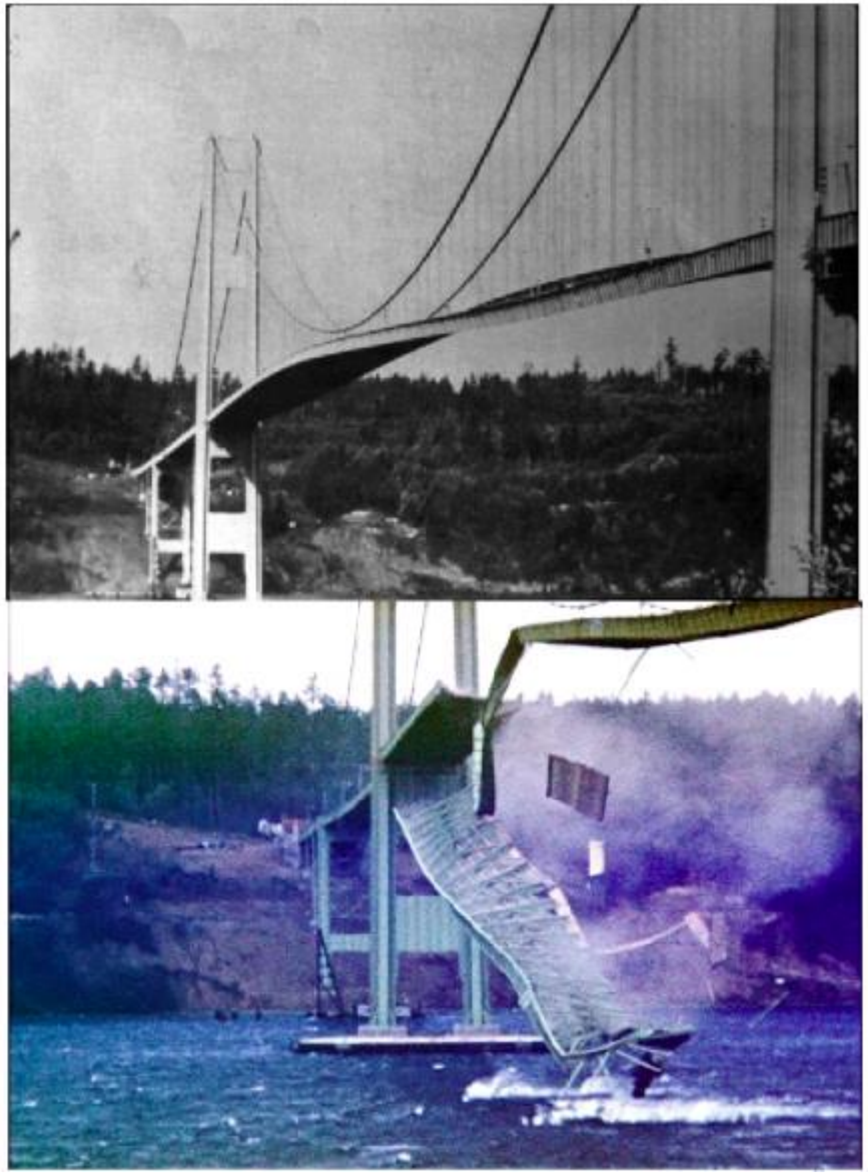
A normal mode is a motion where all parts of the system are moving sinusoidally with the same frequency and in phase.

All observed configurations of a system may be generated from its normal modes.

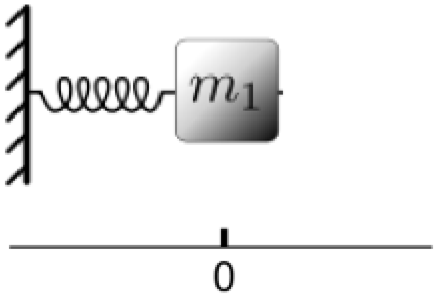
Each normal mode has a characteristic frequency, its eigenvalue.

Normal modes

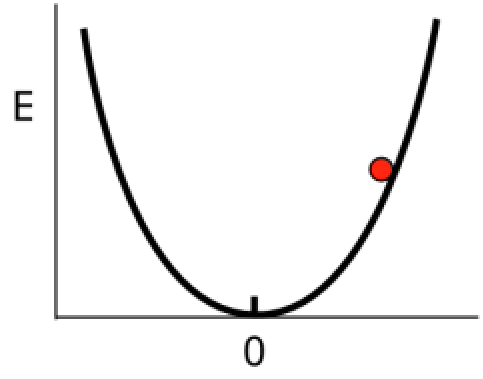
The Tacoma Narrows bridge, opened in 1940, had a normal mode that resonated with a 40 mph wind, until it broke.



Simple harmonic motion



$$\mathbf{F} = -k\mathbf{x},$$

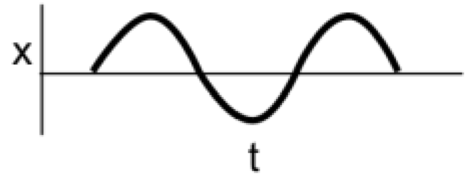


$y=1/2\ x^2$

Imagine a Hooke's Law spring, or a parabolic energy function.
The magnitude of a restoring force is proportional to displacement.
Solving the differential equation

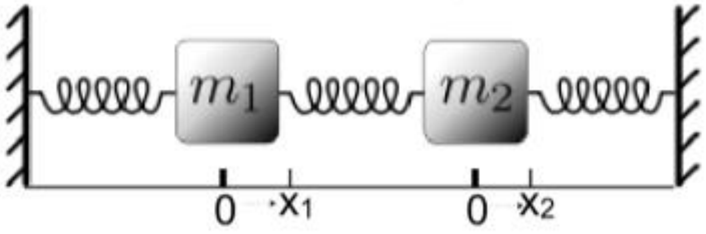
$$F_{net} = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx,$$

Gives a harmonic equation for *x* at time *t* :



$$\mathrm{m}x(t) = A\cos(\nu t) + B\sin(\nu t) = C\cos(\nu t - \phi)$$

Normal modes for coupled oscillators



If we denote acceleration (the second derivative of *x* with respect to time), a form of the equations of motion emerges :

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1) \qquad m\ddot{x}_2 = -kx_2 + k(x_1 - x_2)$$

We assume oscillatory motion so that the complex Fourier series

$$x_1 = C_1 e^{i\nu t} \text{ and } x_2 = C_2 e^{i\nu t}$$

can be substituted into the equations of motion, which gives:

$$-\nu^2 m C_1 e^{i\nu t} = -2kC_1 e^{i\nu t} + \mathrm{k}C_2 e^{i\nu t}$$
$$-\nu^2 m C_2 e^{i\nu t} = kC_1 e^{i\nu t} - 2\mathrm{k}C_2 e^{i\nu t}$$

Since the exponential factor is common to all terms, we omit and simplify:

$$x_1 = C_1 e^{i\nu t} \text{ and } x_2 = C_2 e^{i\nu t}$$

can be substituted into the equations of motion, which gives:

$$(\nu^2 m - 2k)C_1 + \mathrm{k}C_2 = 0$$
$$kC_1 + (\nu^2 m - 2\mathrm{k})C_2 = 0$$

and in matrix representation:

$$\begin{pmatrix} \nu^2 m - 2k & k \\ k & \nu^2 m - 2k \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

Solve by taking the determinant of the matrix.

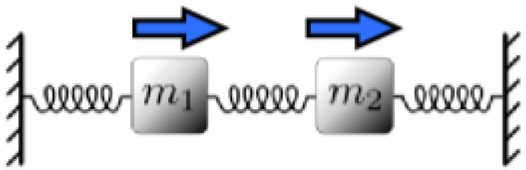
$$(\nu^2 m - 2k)^2 - k^2 = 0$$

Which has two solutions:

$$\nu_1 = \sqrt{\frac{k}{m}}; \qquad \nu_2 = \sqrt{\frac{3k}{m}}$$

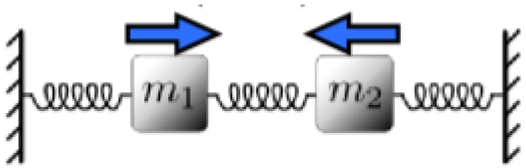
If we substitute ν_1 into the matrix and solve for (C_1, C_2), we get (1, 1). If we substitute ν_2 into the matrix and solve for (C_1, C_2), we get (1, -1). These vectors are eigenvectors, and the frequencies ν are eigenvalues.

The first normal mode (1, 1) is the low frequency mode ν_1



which corresponds to both masses moving in the same direction at the same time.

The second normal mode (1, -1) is the high frequency mode ν_2



which corresponds to the masses moving in the opposite directions, while the center of mass remains stationary.

Characteristics of normal mode motion

- Each normal mode acts like a simple harmonic oscillator
- These fixed frequencies of a system are known as its natural frequencies or resonant frequencies
- The center of mass doesn’t move
- Normal modes are orthogonal to each other. They resonate independently
- The most general motion of a system is a superposition of its normal modes

Relevance to atmosphere (and ocean)

- The ocean and atmosphere are thin sheets of fluid where the horizontal extent is much larger than their vertical extent.
- Most motion energetics is in horizontal mode. But there can be a vertical mode contribution for some situations.
- First employed by Laplace for tidal equations
- In layered depictions of a fluid, there will be a discrete set of modes. In the continuously stratified atmosphere, in principle, there is an infinite numbers of modes. In practice, they will truncate to the number of wave solutions that characterize most of the physics.
- This allows us to use independent complex Fourier wave solutions:

One dimension: $\psi(x,t) = \hat{\psi}e^{i(kx-\nu t)}$

Two dimensions: $\psi(x,y,t) = \hat{\psi}e^{i(kx+ly-\nu t)}$

Three dimensions: $\psi(x,y,z,t) = \hat{\psi}e^{i(kx+ly+mz-\nu t)}$

where *k*, *l*, and *m* are the x, y, and z wavenumbers. A wavenumber is simply 2π/wavelength for each x, y, and z component. *v* is the frequency.

Wave solution strategy

1. Isolate a physical phenomenon represented by a wave solution. In meteorology, these will be ***sound waves, gravity waves, inertial waves, Rossby waves,*** and ***Kelvin waves***. There are variants and combinations of these waves as well.

2. Linearize the equations. We will discuss this procedure soon.

3. Assume a complex Fourier wave solution, and substitute into the linearized equations. Most equations will have a form such as:

$$\alpha\psi^n + \varepsilon\psi^{n-1} + \cdots \mu\psi' = 0$$

where the n derivatives of ψ could represent time derivatives or spatial derivatives for different combinations of wind components, pressure, density, temperature, potential temperature, height, vorticity, etc.

Note that substituting $\psi(x, y, z, t) = \hat{\psi}e^{i(kx+ly+mz-\nu t)}$ into the time or spatial derivatives will result in $\hat{\psi}$, *i*, and $e^{i(kx+ly+mz-\nu t)}$ either cancelling or being pulled outside of an expression as a constant. All that will be left to solve are algebraic combinations of wavenumbers, frequencies, and the constants (generically written above as α, ε, μ…). This is why the concise version of the Fourier series is clever to use. This will become clearer in some examples.

4. Solve for frequency ν, known as the *dispersion relationship*. Then determine *phase speed c,* and *group velocity* c_g, done as:

$$c = \frac{\nu}{wavenumber}; \qquad c_g = \frac{\partial \nu}{\partial(wavenumber)}$$

for the appropriate x, y, or z direction (in other words, *k*, *l*, or *m*.)