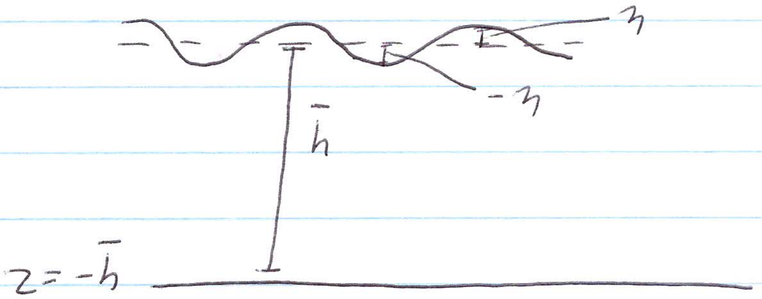
Dispersion relationship for waves at a discontinuity.

This is applicable to ocean waves.



$\eta \approx h'$

At η, $p' = \bar{\rho} g h'$

$w = 0,\ \frac{\partial p'}{\partial z} = 0$ at bottom

Otherwise $w(h) = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$

The pertinent equations are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Linearize as $u = \bar{u} + u'$; $w = \bar{w} + w'$; $p = \bar{p} + p'$; $h = \bar{h} + h'$

Note that $-\frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial z} + g = 0$; $\bar{p} = -\bar{\rho} g z$

One obtains

(I) $$\frac{\partial u'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x}$$

(II) $$\frac{\partial w'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z}$$

(III) $$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

We want to obtain a single equation for p'

$$\frac{\partial}{\partial x}\text{(I)} + \frac{\partial}{\partial y}\text{(II)}$$

which gives

$$\underbrace{\frac{\partial^2 u'}{\partial t \partial x} + \frac{\partial^2 w'}{\partial t \partial z}}_{= \frac{\partial}{\partial t}\left(\underbrace{\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}}_{=0 \text{ from (III)}}\right)} = -\frac{1}{\bar{\rho}}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2}\right)$$