Perturbation Method

- With this method, all filed variables are separated into two parts: (a) a basic state part and (b) a deviation from the basic state:

$$u(x,t) = \overline{u} + u'(x,t)$$

Basic state
(time and zonal mean)

Perturbation

$$u\frac{\partial u}{\partial x} = \left(\overline{u} + u'\right)\frac{\partial}{\partial x}\left(\overline{u} + u'\right) = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

Basic Assumptions

- ***Assumptions 1***: : the basic state variables must themselves satisfy the governing equations when the perturbations are set to zero.
- ***Assumptions 2:*** the perturbation fields must be small enough so that all terms in the governing equations that involve products of the perturbations can be neglected.

$$|u'/\overline{u}| \ll 1$$

neglected

$$u\frac{\partial u}{\partial x} = (\overline{u} + u')\frac{\partial}{\partial x}(\overline{u} + u') = \overline{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x}$$

Example

Original equations

$$\frac{du}{dt} + \alpha \frac{\partial p}{\partial x} = 0$$

$$\frac{dw}{dt} + \alpha \frac{\partial p}{\partial z} + g = 0$$

$$\alpha \frac{dp}{dt} + p\gamma \frac{d\alpha}{dt} = 0$$

$$\alpha \boldsymbol{\nabla} \cdot \mathbf{V} - \frac{d\alpha}{dt} = 0$$

Linearized equations

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + \bar{\alpha} \frac{\partial p'}{\partial x} = 0$$

$$\delta_1 \left(\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} \right) + \bar{\alpha} \frac{\partial p'}{\partial z} - \frac{g\alpha'}{\bar{\alpha}} = 0$$

$$\bar{\alpha} \left(\frac{\partial p'}{\partial t} + U \frac{\partial p'}{\partial x} \right) - gw' + \bar{p}\gamma \left(\frac{\partial \alpha'}{\partial t} + U \frac{\partial \alpha'}{\partial x} + w' \frac{\partial \bar{\alpha}}{\partial z} \right) = 0$$

$$\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} \right) \bar{\alpha} - \delta_2 \left(\frac{\partial \alpha'}{\partial t} + U \frac{\partial \alpha'}{\partial x} \right) - w' \frac{\partial \bar{\alpha}}{\partial z} = 0$$

Purpose of Perturbation Method

- If terms that are products of the perturbation variables are neglected, the nonlinear governing equations are reduced to linear differential equations in the perturbation variables in which the basic state variables are specified coefficients.
- These equations can then be solved by standard methods to determine the character and structure of the perturbations in terms of the known basic state.
- For equations with constant coefficients the solutions are sinusoidal or exponential in character.
- Solution of perturbation equations then determines such characteristics as the propagation speed, vertical structure, and conditions for growth or decay of the waves.
- The perturbation technique is especially useful in studying the stability of a given basic state flow with respect to small superposed perturbations.