Sound (acoustic) waves

The pertinent equations are:

(I) $\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

(II) $\frac{D\rho}{Dt} = -\rho\frac{\partial u}{\partial x}$ or $\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{\partial u}{\partial x}$ or $\frac{D\ln\rho}{Dt} = -\frac{\partial u}{\partial x}$

(III) $C_p\frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = 0$

(III) can be written as

$$C_v\frac{D\ln p}{Dt} - C_p\frac{D\ln\rho}{Dt} = 0$$

Let $\gamma = \frac{C_p}{C_v}$

(IV) $\frac{1}{\gamma}\frac{D\ln p}{Dt} - \frac{D\ln\rho}{Dt} = 0$ This has removed T as a variable

Substitute (II) into (IV), multiply by γp

(V) $\frac{Dp}{Dt} + \gamma p\frac{\partial u}{\partial x} = 0$ This has removed ρ as a variable in two equations

Linearize (I) and (V) as $u = \bar{u} + u'$; $p = \bar{p} + p'$; $\rho = \bar{\rho} + \rho'$

Expand total derivative $\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + u\frac{\partial(\,)}{\partial x}$

$$\cancel{\frac{\partial \bar{u}}{\partial t}}^{\to 0} + \frac{\partial u'}{\partial t} + \bar{u}\cancel{\frac{\partial \bar{u}}{\partial x}}^{\to 0} + \bar{u}\frac{\partial u'}{\partial x} + u'\cancel{\frac{\partial \bar{u}}{\partial x}}^{\to 0} + \cancel{u'\frac{\partial u'}{\partial x}}^{\text{small, neglect}} = \frac{-1}{(\bar{\rho}+\rho')}\cancel{\frac{\partial \bar{p}}{\partial x}}^{\to 0} - \frac{1}{(\bar{\rho}+\rho')}\frac{\partial p'}{\partial x}$$

Note from

$$\frac{1}{\bar{\rho}}\,\frac{1}{[1+\rho'/\bar{\rho}]} \approx \frac{1}{\bar{\rho}}\left[1 - \cancel{\frac{\rho'}{\bar{\rho}}}^{\text{small, neglect}}\right] \approx \frac{1}{\bar{\rho}}$$

$$\cancel{\frac{\partial \bar{p}}{\partial t}}^{\to 0} + \frac{\partial p'}{\partial t} + \bar{u}\cancel{\frac{\partial \bar{p}}{\partial x}}^{\to 0} + \bar{u}\frac{\partial p'}{\partial x} + u'\cancel{\frac{\partial \bar{p}}{\partial x}}^{\to 0} + \cancel{u'\frac{\partial p'}{\partial x}}^{\text{small, neglect}}$$

$$+ \gamma\bar{p}\cancel{\frac{\partial \bar{u}}{\partial x}}^{\to 0} + \gamma p'\cancel{\frac{\partial \bar{u}}{\partial x}}^{\to 0} + \gamma\bar{p}\frac{\partial u'}{\partial x} + \cancel{\gamma p'\frac{\partial u'}{\partial x}}^{\text{small, neglect}} = 0$$

The final linearized equations are

(VI) $$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0$$

(VII) $$\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x} + \gamma\bar{p}\frac{\partial u'}{\partial x} = 0$$